SOLUTIONS

OF THE

CAMBRIDGE PROBLEMS

FOR THE YEARS 1840, 1841

BY

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PREFACE.

The present work is intended as an aid to the student, in the preparation for the Senate-House, during his last Term. Many of the solutions might have been given in a more elaborate form, and this was the original design of the author, but as every moment is valuable to the student near the time of his final examination, and all the author could expect or wish, would be a hasty perusal of the work, it was deemed better to be as concise as possible, consistently with the design of bringing prominently forward general principles applicable to a class of problems, in preference to using artifices which, though they might shorten the process in particular instances, would not be equally beneficial to the student in preparing him to meet future cases.

The problems of the last two years have been selected, both because they offer every possible variety, and as they will give a better

idea of the present character of the Senate-House Examinations.

The author had prepared for publication the solutions of the Geometrical Problems, whether of two or three dimensions, according to a new method of Geometry, which has been employed successfully on the continent by M. Poncelet, and M. Chasles, but has not appeared in any Cambridge work. It was thought, however, advisable to comprise these, together with a short account of the Geometry, in a tract, which the author hopes soon to lay before the student.

The author takes this opportunity of returning his thanks to the Examiners, and his friends, for their advice and assistance during the preparation of the work.

Saint John's College,
Sept. 18, 1841.

MR THURTELL'S PAPER.

JANUARY 7, 1840.

Problem I.

A fraction whose denominator is less than ten, when reduced to the decimal form, cannot contain the figure 9 *in its decimal part. Are any other digits excluded in particular cases?*

Let $\frac{p}{q}$ be the fraction in which $p < q < 10$. Then in the process of the division of $10^m p$ by q, the remainder must be always less than the divisor, and the greatest remainder will be $q-1$: the succeeding quotient will therefore be $\frac{10(q-1)}{q} = 10 - \frac{10}{q}$, which since q is an integer less than 10, can never give 9 as a result; and this follows *à fortiori* when the remainder is less than the assumed one.

In different particular cases different digits are excluded. Thus when $q = 3$, the only admissible remainders are 1, 2, and these give for quotients 3 and 6. Similarly when

$q = 4$ the excluded digits are 1, 3, 4, 6, 8, 9,

$q = 5$ 1, 3, 5, 7, 9,

$q = 6$ 2, 4, 7, 9,

$q = 7$ 3, 6, 9,

$q = 8$ 4, 9,

$q = 9$ .... 9.

PROBLEM II.

Two circles intersect in A *and* B: AD, AD′ *are diameters*: AC, AC′ *are chords, each of which touches the circle of which it is not a chord: the line* AEE′ *bisects the angle* DAD′ *and cuts the circles in* E *and* E′: *then the common tangent to the circles is a mean proportional between the chords* DE, D′E′: *and their common chord* (AB) *is a mean proportional between the chords* BC, BC′.

Fig. 1. Let O, O' be the centres of the circles: r, R their radii. Then since DBA, $D'BA$ are each right angles: DBD' is a straight line and $= 2\,OO' = 2d$, suppose.

$$\text{Then } DE.D'E' = 4Rr\sin^2 DAE$$

$$= 2Rr\left(1 - \frac{r^2 + R^2 - d^2}{2rR}\right)$$

$$= d^2 - (R-r)^2 = (\text{common tang.})^2.$$

$$\text{Also,}\qquad ACB = ADB = \frac{\pi}{2} - DAB$$

$$= DAC' - DAB$$

$$= BAC'.$$

$$\text{Similarly,}\qquad AC'B = BAC;$$

therefore the triangles ACB, $AC'B$ are similar,

$$\text{and}\quad \frac{CB}{AB} = \frac{AB}{BC'},$$

$$\text{or}\quad CB.BC' = AB^2.$$

Problem III.

Three indefinite straight lines intersect in A, B, C: *any other straight line cuts* AB, BC, CA *in* C′, A′, B′ *respectively: then* AB′ . BC′ . CA′ = A′B . B′C . C′A, *and the product of the areas* A′BC′, B′CA′, C′AB′

$$=(A'B'.B'C'.C'A' \sin A' \sin B' \sin C')^2 \div 8 \sin A \sin B \sin C.$$

Fig. 2. We have by trigonometry,

$$\frac{AB'}{C'A} = \frac{\sin C'}{\sin B'}, \qquad \frac{CA'}{B'C} = \frac{\sin B'}{\sin A'}, \qquad \frac{BC'}{A'B} = \frac{\sin A'}{\sin C'};$$

$$\therefore \frac{AB'.BC'.CA'}{C'A.A'B.B'C} = 1,$$

and 8 product of the areas

$$= A'C' \ C'B \sin C' . B'A' . A'C \sin A' . B'C' . B'A \sin B'$$

$$= A'C'^2 . \frac{\sin A' \sin C'}{\sin B} . A'B'^2 . \frac{\sin B' \sin A'}{\sin C} \ B'C'^2 . \frac{\sin C' \sin B'}{\sin A}$$

$$= (A'C' . A'B' . B'C' \sin A' \sin B' \sin C')^2 \div \sin A \sin B \sin C.$$

Problem IV.

If a point C′ *be taken in any one (as* AB) *of three indefinite straight lines that intersect in* A, B *and* C, *and lines (as* C′, B′, A′) *be drawn from* C′ *cutting* AC, BC *(as in* B′, A′), *then all the intersections of each pair of lines (as* BB′, AA′) *drawn from* B *and* A *to the points of section* (B′, A′) *lie in a line that passes through* C.

Fig. 3. Let Bx, By be axes of co-ordinates:

$$AB = c, \quad BC = a, \quad AC = b, \quad BC' = k, \quad BA' = u.$$

$$\left.\begin{aligned}\text{The equation to } A'C' \text{ is } \frac{x}{z}+\frac{y}{k}&=1\\ \text{.................. } AC \text{ is } \frac{x}{a}+\frac{y}{c}&=1\end{aligned}\right\}.$$

Eliminating between these equations, we have

$$\left.\begin{aligned}B'M&=\frac{z-a}{\dfrac{z}{k}-\dfrac{a}{c}}\\ BM&=\frac{k-c}{\dfrac{k}{z}-\dfrac{c}{a}}\end{aligned}\right\}.$$

The equation to BB' is therefore

$$y\,.\,BM=x\,.\,B'M,$$

$$\left.\begin{aligned}\text{or}\quad y\,.\,\frac{k-c}{\dfrac{k}{z}-\dfrac{c}{a}}&=x\,.\,\frac{z-a}{\dfrac{z}{k}-\dfrac{a}{c}}\\ \text{The equation to } AA' \text{ is } \frac{x}{z}+\frac{y}{c}&=1\end{aligned}\right\}.$$

The elimination of z between these two equations, which corresponds to the intersection of BB' with AA', gives

$$\frac{y}{ck}+\frac{x}{(2k-c)a}=\frac{1}{2k-c},$$

and making $y=0$ we have $x=a$, so that the locus of these intersections is a line passing through C.

Problem V.

Two straight lines which coincide in their initial position, revolve uniformly with different angular velocities about two fixed points: find the locus of their points of intersection and trace the curve when the angular velocity of one line is twice that of the other.

If these lines be one pair only of several arranged like the spokes of a wheel about the fixed points, find, for any given position of the first pair, the curve passing through the simultaneous intersections of all the pairs, formed by taking lines that have any equal angular distances from the first.

Fig. 4. Let A, B be the fixed points; AB the initial position of the lines, of which the angular velocity of one is n times that of the other. Let AP, BP be the lines in any position.

$$AP = r, \quad PAB = \theta, \quad AB = a.$$

Then
$$\frac{AP}{AB} = \frac{\sin ABP}{\sin APB},$$

or
$$\frac{r}{a} = \frac{\sin n\theta}{\sin (n-1)\theta}, \text{ the equation required.}$$

When
$$n = 2, \quad r = \frac{a \sin 2\theta}{\sin\theta} = 2a\cos\theta,$$

which is the polar equation to a circle, of which B is the centre, and AB the radius.

Again, let AQ, BQ be any other pair of lines, making angles $k\gamma$, $k\beta$ with AP, BP respectively; and let $PAB = \alpha$.

Then if $AQ = \rho$, $QAB = \theta = \alpha + k\gamma$,

we have
$$\frac{\rho}{a} = \frac{\sin ABQ}{\sin AQB},$$

$$\text{or} \quad \rho = a \frac{\sin(n\alpha + k\beta)}{\sin\{(n-1)\alpha + k(\beta-\gamma)\}},$$

$$\text{and} \quad k = \frac{\theta - a}{\gamma};$$

$$\therefore \rho \sin\left\{(n-1)\alpha + \frac{\beta-\gamma}{\gamma}(\theta - a)\right\} = a \sin\left\{n\alpha + \frac{\beta}{\gamma}(\theta - a)\right\},$$

which is the equation to the locus.

Problem VI.

If the axes of two equal cylinders of radius (a) *intersect at an angle* (α) *the volume common to both* $= \frac{16}{3}\frac{a^3}{\sin\alpha}$, *the surface common to both* $= 16\frac{a^2}{\sin\alpha}$.

Fig. 5. Let the axes of x and z be the axes of the cylinders, and the axis of y perpendicular to the plane containing them. Then the sections by the planes yz, xy will be ellipses whose semimajor and semiminor axes are $\frac{a}{\sin\alpha}$ and a respectively.

Now if V be the volume of a portion of the solid contained between two planes parallel to yz, xz at distances x, y from them,

$$\text{then} \quad \delta V = z \sin\alpha \,.\, \delta x \delta y \text{ ultimately};$$

this being the volume of a small prism of which $\delta x \delta y$ is the area of the base, and $z \sin\alpha$ the altitude;

$$\therefore \quad V = \sin\alpha \int_x \int_y z.$$

$$\text{Now} \qquad \frac{z^2 \sin^2\alpha}{a^2} + \frac{y^2}{a^2} = 1,$$

$$\text{and} \quad \frac{x^2 \sin^2 \alpha}{a^2} + \frac{y^2}{a^2} = 1,$$

are the equations to the two cylinders;

$$\therefore \; V = \int_x \int_y \sqrt{a^2 - y^2}.$$

The limits of x being $x = 0$, and $x = \dfrac{a}{\sin \alpha} \sqrt{1 - \dfrac{y^2}{a^2}}$

.... y ...... $y = 0$, $y = a$;

$$\therefore \; V = \int_y \frac{a^2 y - \frac{1}{3} y^3}{\sin \alpha} = \frac{2}{3} \frac{a^3}{\sin \alpha},$$

$$\text{and whole volume} = \frac{16}{3} \frac{a^3}{\sin \alpha}.$$

Again, if δS be an element of the common surface,

$$\delta S = \delta x \delta y \sqrt{1 + (d_x z)^2 \sin^2 \alpha + (d_y z)^2 \sin^2 \alpha},$$

$$\text{and } S = \int_x \int_y \sqrt{1 + (d_y z)^2 \sin^2 \alpha}, \text{ (since } d_x z = 0)$$

$$= \int_y x \sqrt{1 + \frac{y^2 \sin^2 \alpha}{\sin^2 \alpha \, (a^2 - y^2)}}$$

$$= \int_y \cdot \frac{a}{\sin \alpha},$$

and surface $CDPB = \dfrac{a^2}{\sin \alpha}$, and this also is the area of surface $BPDA$;

$\therefore$ whole surface common to both cylinders

$$= \frac{16 a^2}{\sin \alpha}.$$

Problem VII.

A thin hemispherical bowl of given weight, partly filled with fluid, is placed with its axis vertical upon the highest point of a sphere: find the nature of its equilibrium as respects stability.

Fig. 6. Let R, r be the radii of the lower and upper spheres; ρ the radius of the plane of floatation of the fluid; V its volume; M, $\frac{M}{n}$ the masses of the bowl and fluid. Suppose the bowl to be displaced in a vertical plane through a very small angle, so that they are in contact at R; A, B having been their original points of contact, O, C the centres of the spheres.

$AOR = \phi$, $BCR = \theta$; and let the vertical through R meet the axis of the bowl in K: then the equilibrium will be stable or unstable according as the perpendicular from R on the resultant of the forces acting on the bowl falls on the side of R nearer to or farther from B; that is, if L be the point of application of this resultant, according as AL is $<$ or $>$ AK.

Now
$$R\theta = r\phi,$$

and
$$\frac{OK}{r} = \frac{\sin\theta}{\sin(\theta + \phi)} = \frac{r}{R + r}, \text{ approximately;}$$

$$AK = \frac{R \,.\, r}{R + r}.$$

Again, remembering that the weight of the fluid acts downwards through the metacentre,

$$M \,.\, \frac{r}{2} + \frac{M}{n} \,.\, \left(h + \frac{\pi\rho^4}{4V}\right) = AL \,.\, \frac{n+1}{n} \,.\, M.$$

Therefore the equilibrium will be stable or unstable, according as

$$\frac{nr}{2} + h + \frac{\pi\rho^4}{4V} \text{ is } < \text{ or } > \frac{(n+1)Rr}{R+r};$$

h being the distance of the centre of gravity of the fluid from A, when in its original position.

PROBLEM VIII.

The highest point of the wheel of a carriage rolling on a horizontal road, moves twice as fast as each of two points in the rim, whose distance from the ground is half the radius of the wheel.

Find the rate at which the carriage is travelling, when the dirt thrown from the rim of the wheel to the greatest height reaches a given level. Explain the two roots given by the resulting equation. If the velocity of the carriage be less than that due to a height equal to half the radius of the wheel, what is the greatest height to which the dirt is thrown?

Fig. 7. Let Q, Q' be two points in the rim of the wheel, such that the foot of their common ordinate bisects the radius. Then for an instant the wheel revolves about B with an angular velocity ω;

$$\left.\begin{aligned}\text{and velocity of } Q &= \omega \cdot QB \\ \dots\dots\dots\dots\dots\ A &= \omega \cdot AB\end{aligned}\right\};$$

$$\therefore \frac{\text{vel. of } Q}{\text{vel. of } A} = \frac{QB}{AB} = \sqrt{\frac{BN}{AB}} = \frac{1}{2};$$

$$\therefore \text{ vel. of } Q \text{ or } Q' = \frac{1}{2} \cdot \text{vel. of } A.$$

Again, let P be a point in the wheel which is moving in the direction PA, or in a tangent to its cycloidal path.

$$BM = x, \quad AB = 2a, \quad v = \text{vel. of the carriage.}$$

The equation to the path of any particle projected from the wheel at P is

$$Y = X \tan APM - \frac{g X^2}{2\omega^2 . BP^2} (1 + \tan^2 \alpha),$$

and the height above PM is a maximum when $d_X Y = 0$.

This gives maximum height $= \dfrac{\omega^2 . BP^2}{2g} . \dfrac{AM^2}{AP^2}$

$$= \frac{\omega^2}{2g} (2ax - x^2).$$

And we are to find the highest point of all the parabolic paths.

Hence $\qquad x + \dfrac{\omega^2}{2g} (2ax - x^2) = \text{maximum};$

$$\therefore 1 + \frac{\omega^2}{g} (a - x) = 0,$$

$$x = a + \frac{g}{\omega^2};$$

$$\therefore \text{greatest height} = a + \frac{g}{\omega^2} + \frac{\omega^2}{2g} \left(a + \frac{g}{\omega^2}\right) \left(a - \frac{g}{\omega^2}\right)$$

$$a + \frac{g}{\omega^2} + \frac{a^2 \omega^2}{2g} - \frac{g}{2\omega^2}.$$

and if v be the velocity of the carriage,

$$\text{greatest height} = a + \frac{g a^2}{2 v^2} + \frac{v^2}{2g} = d, \text{ suppose};$$

$$\therefore v^4 - 2g(d - a) . v^2 + g^2 a^2 = 0;$$

$$\therefore v^2 = g \left\{ d - a \pm \sqrt{(d - a)^2 - a^2} \right\}.$$

Now $\qquad x = a + \dfrac{g a^2}{v^2} = d \mp \sqrt{d^2 - 2ad}.$

Now x cannot be greater than $2a$, and d cannot be less than $2a$. Hence $x < d$, and we have

$$x = d - \sqrt{d^2 - 2ad},$$

$$\text{and} \quad v^2 = g(d - a + \sqrt{d^2 - 2ad})$$

$$\therefore \pm v = \sqrt{\frac{gd}{2}} + \sqrt{\frac{gd}{2} - ga}. \quad (1)$$

This expresses that there is another point on the opposite side of the wheel, from which, if the carriage moved with the same velocity and in the opposite direction, the dirt would be projected to the same height.

We observe also from the above equations that as v diminishes x increases; and when

$$v = \sqrt{ga}, \quad x = 2a, \quad d = 2a.$$

Now equation (1) expresses that when $d < 2a$ v is imaginary, or in other words, there is no value of v which can make $d < 2a$. Hence when $v < \sqrt{ga}$, the greatest height is always the highest point of the wheel, which is also the point from which the dirt is thrown.

Problem IX.

Demonstrate formulæ for calculating the time occupied by the disk of the moon in any position in the heavens, whilst crossing first the vertical, and secondly the horizontal wire of a telescope directed to it.

Fig. 8. Let F be the pole of the heavens, considered as a sphere of which C the observer's eye is the centre; PQ the path of the moon's centre, whilst crossing the vertical wire of the telescope; δ her declination; AB the portion of a great circle corresponding to PQ: m the moon's relative motion in right ascension; Δ her apparent diameter.

Then $$\frac{QP}{AB} = \sin FP = \cos\delta,$$

and time in crossing the vertical wire

$$= \frac{AB}{m} = \frac{\Delta \sec\delta}{m}.$$

So if l be the angle subtended at the eye by the horizontal wire, time in crossing the horizontal wire

$$= \frac{(\Delta + l)\sec\delta}{m}.$$

PROBLEM X.

Assuming the known theorem

$$\mathrm{a} = \mathrm{b}\,cos\,(\mathrm{ab}) + \mathrm{c}\,cos\,(\mathrm{ac}) + \mathrm{d}\,cos\,(\mathrm{ad}) + \&c.$$

where a, b, c, *&c. are the sides of a plane polygon,* (ab) *the angle between* a *and* b, *&c., prove that*

$$\Sigma(\mathrm{a}^2) = 2\,\Sigma\,\{\mathrm{ab}\,cos\,(\mathrm{ab})\}.$$

Since $a = b\cos(ab) + c\cos(ac) + d\cos(ad) + \&c.,$

we have $a^2 = ab\cos(ab) + ac\cos(ac) + ad\cos(ad) + \&c.$

Similarly $b^2 = ba\cos(ba) + bc\cos(bc) + bd\cos(bd) + \&c.$

and $c^2 = ca\cos(ca) + cb\cos(cb) + cd\cos(cd) + \&c.$

$d^2 = da\cos(da) + db\cos(db) + dc\cos(dc) + \&c.$

$\&c. = \&c.;$

$$\therefore\ a^2 + b^2 + c^2 + d^2 + \&c.$$

$$= 2ab\cos(ab) + 2bc\cos(bc) + 2ac\cos(ac)$$
$$+ 2bd\cos(bd) + 2cd\cos(cd)$$
$$+ 2ad\cos(ad) + \&c.$$

or $\Sigma(a^2) = 2\,\Sigma\,\{ab\cos(ab)\}.$

Problem XI.

If the instrument be perfect by which in surveying a series of points are laid down, from the observed angles which the distances between three stations in the same plane, whose positions are accurately known, subtend at each point, shew how an error in one of the observed angles affects the position of the point to be laid down.

Fig. 8 *bis*. Let A, C, B be the fixed stations: P the point laid down: $CP = r$, $ACP = \phi$, $APC = \alpha$, $BPC = \beta$, $AC = b$, $BC = a$, $AB = c$.

$$\text{Then } \frac{r}{b} = \frac{\sin(\alpha + \phi)}{\sin\alpha}, \qquad \frac{r}{a} = \frac{\sin(C + \beta - \phi)}{\sin\beta};$$

$$\therefore \frac{\delta r}{b} = \delta\phi \cdot \frac{\cos(\alpha + \phi)}{\sin\alpha} - \delta\alpha \cdot \frac{\sin\phi}{\sin^2\alpha},$$

$$\frac{\delta r}{a} = -\delta\phi \cdot \frac{\cos(C + \beta - \phi)}{\sin\beta};$$

$$\frac{\sin\alpha}{\cos(\alpha + \phi)}\left(\frac{\delta r}{b} + \delta\alpha \cdot \frac{\sin\phi}{\sin^2\alpha}\right) = -\frac{\delta r}{a} \cdot \frac{\sin\beta}{\cos(C - \phi + \beta)};$$

$$\delta r\left\{\frac{1}{a} \cdot \frac{\sin\beta}{\cos(C-\phi+\beta)} + \frac{1}{b} \cdot \frac{\sin\alpha}{\cos(\alpha+\phi)}\right\} = -\frac{\sin\phi}{\cos(\alpha+\phi)\sin\alpha}\,\delta\alpha,$$

$$\text{or } k\delta r = -\frac{\sin\phi}{\cos(\alpha + \phi)\sin\alpha}\,\delta\alpha,$$

$$\text{and } kr\delta\phi = \tan(C + \beta - \phi) \cdot \frac{\sin\phi}{\cos(\alpha + \phi)\sin\alpha}\,\delta\alpha;$$

$$\therefore k^2\left\{(\delta r)^2 + (r\delta\phi)^2\right\} = \frac{\sin^2\phi}{\cos^2(\alpha+\phi)\sin^2\alpha} \cdot \sec^2(C+\beta-\phi)\,(\delta\alpha)^2;$$

$$\therefore \frac{\text{displacement}}{\delta a} = \frac{\sin\phi \div \sin a}{\frac{1}{a}\sin\beta\cos(a+\phi) + \frac{1}{b}\sin a\cos(C-\phi+\beta)}$$

$$= \frac{r\sin\phi}{\sin a\sin(C+a+\beta)}$$

$$= \frac{b\sin\phi\sin(a+\phi)}{\sin^2 a\sin(C+a+\beta)}.$$

Now if ϕ be found from the equation

$$\frac{b\sin(a+\phi)}{\sin a} = \frac{a\sin(C+\beta-\phi)}{\sin\beta}$$

and its value substituted in the above equation, we have

$$\frac{\text{displacement}}{\delta a} = \frac{ab\{a\sin(C+\beta) - b\sin\beta\}}{a^2\sin^2 a + 2ab\sin a\sin\beta\cos(a+\beta+C) + b^2\sin^2\beta}.$$

Now the denominator is zero when P lies in the circle circumscribing the triangle ABC, and the numerator is ab. Hence the displacement will be considerable when the point to be laid down is near the circumference of this circle.

Problem XII.

The tangents to the interior of two concentric and similar curves of the second order whose axes are coincident, cut off from the exterior curve equal areas.

Fig. 9. Let C be the common centre, CAB the common axis of the two curves, whose semiaxes are a_1, b_1, a_2, b_2, connected by the relation $\frac{b_1}{a_1} = \frac{b_2}{a_2}$, since they are similar.

Let KL be the tangent to the interior of the two curves: Cp conjugate to CL: let CL meet the exterior curve in D. Then

$$\tan\phi = \pm\frac{b_1^2}{a_1^2}.\frac{CM}{DM} = \pm\frac{b_2^2}{a_2^2}.\frac{CN}{LN}.$$

Hence the tangent at D is parallel to KL, and therefore parallel to CP, which is therefore conjugate to CD: and the tangent being an ordinate to the exterior curve, is bisected in L.

Let $\qquad CP = b, \qquad CD = a,$

and let x, y be co-ordinates of any point in the exterior curve referred to CP, CD as oblique axes: $PCD = \alpha$.

Then area $DKL = \dfrac{b\sin\alpha}{a}\,{}_{CL}\!\int_x^{CD}\sqrt{a^2 - x^2}.(\pm 1)^{\frac{1}{2}}$

$$= \frac{b\sin\alpha}{a}\left\{x\sqrt{a^2 - x^2}.(\pm 1)^{\frac{1}{2}} + a^2\sin^{-1}\frac{x}{a}\right.$$

$$\left.\text{or} \quad - a^2\log_e(x + \sqrt{x^2 - a^2})\right\}$$

between the limits $x = CL$, $x = a$.

Now if $LCA = \theta$,

$$\frac{a_1^2 b_1^2}{a_1^2\sin^2\theta + b_1^2\cos^2\theta}, \qquad CL^2 = \frac{a_2^2 b_2^2}{a_2^2\sin^2\theta + b_2^2\cos^2\theta},$$

$$\text{and} \quad \frac{b_1}{} = \frac{b_2}{}:$$

$$CL$$

Hence area DKL

$$ab\sin\alpha\left(\frac{\pi}{2} - \frac{b_1}{b_2}\sqrt{1 - \frac{b_1^2}{b_2^2}} - \sin^{-1}\frac{b_1}{b_2}\right),$$

$$\text{or} \quad = ab\sin\alpha\left\{\frac{b_1}{b_2}\sqrt{\frac{b_1^2}{b_2^2} - 1} - \log_e\left(\frac{b_1}{b_2} + \sqrt{\frac{b_1^2}{b_2^2} - 1}\right)\right\}.$$

Now $ab \sin a =$ constant.

Hence area KDS is constant for all positions of the tangent.

Problem XIII.

A uniform rough cylinder is supported with its axis horizontal, equally upon each of two elastic strings, whose weights are inconsiderable: the strings are equal in all respects and attached to points of the same horizontal plane above the cylinder, so that they hang vertically, and each lies entirely in a plane perpendicular to the axis of the cylinder. Find how much the cylinder descends by the stretching of the strings.

Fig. 10. Let $CP = a$ the radius of the cylinder, CD perpendicular to AC, $ACP = \theta$, $ACQ = \theta + \delta\theta$, μ the ratio of friction to pressure, $t =$ tension of the string at P, $p =$ pressure, $2\omega =$ weight of cylinder, $AP = s_1$, the original value of which was s. The cylinder descends, and each part of the string AD stretches, till it attains such a position that each point of AD is on the eve of motion.

$$\text{Tension at } Q = t + d_\theta t \,.\, \delta\theta \text{ ultimately;}$$

therefore the difference of the tensions on the extremities of the element PQ, the equilibrium of which we are considering, and which we suppose to become rigid, $= d_\theta t \,.\, \delta\theta$ ultimately: hence resolving the forces which act on PQ, in and perpendicular to the tangent, we have

$$\left.\begin{aligned} d_\theta t \,.\, \delta\theta &= \mu p a \delta\theta \\ \text{and } 2t \sin\frac{\delta\theta}{2} &= p a \delta\theta \end{aligned}\right\};$$

$$\therefore\ t = pa, \qquad d_\theta t = \mu p a,$$

$$\frac{d_\theta t}{t} = \mu.$$

Integrating, and remembering that when $\theta = 0$, $t = \omega$, we have

$$t = \omega e^{\mu\theta}.$$

Now let a be the quantity by which an unit of length of the string, and subjected to an unit of tension, would be stretched. Then δs units of length under the action of t units of tension would be stretched by $a\delta s t$: and therefore

$$\delta s_1 - \delta s = \delta s a t,$$

$$\text{or } \delta s_1 = \delta s\,(1 + a\omega\epsilon^{\mu\theta}),$$

$$\text{or } d_\theta s = \frac{a}{1 + a\omega\epsilon^{\mu\theta}},$$

$$s = \frac{a}{\mu}\log_\epsilon\left\{\frac{(a\omega + 1)\,\epsilon^{\mu\theta}}{a\omega\epsilon^{\mu\theta} + 1}\right\}, \quad (1)$$

$$\text{and } \frac{\delta s_1 - \delta s}{\delta s_1} = \frac{\delta s}{\delta s_1}\,a\omega\epsilon^{\mu\theta}$$

$$\frac{a\omega\epsilon^{\mu\theta}}{1 + a\omega\epsilon^{\mu\theta}},$$

$$\text{or } d_\theta(s_1 - s) = \frac{a a\omega\epsilon^{\mu\theta}}{1 + a\omega\epsilon^{\mu\theta}},$$

$$s_1 - s = \frac{a}{\mu}\log_\epsilon(1 + a\omega\epsilon^{\mu\theta}) + C,$$

and when $\theta = 0$, $s_1 = 0$, and from (1) $s = 0$;

$$\therefore\ 0 = \frac{a}{\mu}\log_\epsilon(1 + a\omega) + C;$$

$$\therefore\ s_1 - s = \frac{a}{\mu}\log_\epsilon\left(\frac{1 + a\omega\epsilon^{\mu\theta}}{1 + a\omega}\right),$$

and making $\theta = \frac{\pi}{2}$, we have the whole quantity by which the portion AD has been stretched $= \frac{a}{\mu}\log_\epsilon\left(\frac{1 + a\omega\epsilon^{\frac{\mu\pi}{2}}}{1 + a\omega}\right)$.

Suppose that originally $AO = b$.

Then the quantity by which it has been stretched $= ba\omega$. Therefore the whole string has been stretched by

$$2\left\{ba\omega + \frac{a}{\mu}\log_e\left(\frac{1 + a\omega\epsilon^{\frac{\mu\pi}{2}}}{1 + a\omega}\right)\right\}.$$

Problem XIV.

If θ be the angle between any two lines in space which attract each other with forces $\propto (dist.)^{-2}$, the whole attraction in the direction of the shortest line between them is $= \frac{2\pi}{\sin\theta}$, the mutual attraction of two units of length, collected in centres and separated by the unit of distance, being considered equal to unity.

Fig. 11. Let the shortest line be the axis of z, and the origin its point of bisection; and let the plane parallel to both and perpendicular to the axis of z, be the plane xy, the axis of x being taken to bisect the angle between the projections of the lines on that plane, each of which makes an angle $\frac{1}{2}\theta$ with it.

Let xyz, XYZ be the co-ordinates of any two points in the lines AP, BQ, whose distances from A and B are r, r_1 respectively.

Then $\quad PQ^2 = (x - X)^2 + (y - Y)^2 + AB^2,$

and $\quad r\cos\frac{\theta}{2} = x, \qquad r\sin\frac{\theta}{2} = y$

and $AB = 2c$;

$$r_1\cos\frac{\theta}{2} = X, \qquad r_1\sin\frac{\theta}{2} = -Y.$$

$$\therefore\ PQ^2 = \cos^2\frac{\theta}{2}(r - r_1)^2 + \sin^2\frac{\theta}{2}(r + r_1)^2 + 4c^2$$

$$= r^2 - 2rr_1\cos\theta + r_1^2 + 4c^2.$$

Hence the attraction of an element δr on another being proportional to their product

$$= \frac{\delta r \delta r_1}{(r^2 - 2rr_1\cos\theta + r_1^2 + 4c^2)}.$$

Now if ϕ be the inclination of PQ to the axis of z,

$$2c = PQ\cos\phi;$$

$\therefore$ the attraction in the direction of the shortest line

$$= 2c\int_r\int_{r_1}(r^2 - 2rr_1\cos\theta + r_1^2 + 4c^2)^{-\frac{3}{2}}$$

$$= 2c\int_{r_1}\frac{r - r_1\cos\theta}{r_1^2\sin^2\theta + 4c^2}\,\frac{1}{\sqrt{(r - r_1\cos\theta)^2 + r_1^2\sin^2\theta + 4c^2}},$$

between limits $r = -\infty$, $r = +\infty$;

$$\therefore \text{ the whole attraction} = 4c\int_{r_1}\frac{1}{r_1^2\sin^2\theta + 4c^2}$$

$$= \frac{4c}{\sin^2\theta}\,{}^{-\infty}\!\!\int_{r_1}^{+\infty}\frac{1}{r_1^2 + \dfrac{4c^2}{\sin^2\theta}}$$

$$= \frac{2}{\sin\theta}\tan^{-1}\left(\frac{r_1\sin\theta}{2c}\right)$$

$$= \frac{2}{\sin\theta}\left(\frac{\pi}{2} + \frac{\pi}{2}\right) = \frac{2\pi}{\sin\theta}.$$

Problem XV.

A complete pyramidal pile of equal shot upon a square horizontal base, has four shot in each side of the lowest tier. Find the dimensions of the pyramid that envelops and contains the whole pile.

Fig. 50. Conceive four adjacent shot, whose centres form a square, the shot D being at the corner. Then if another shot, whose centre is E, lie on these four, the line DE will evidently be parallel to an edge of the required pyramid; DF will be $\frac{1}{2}$ the diagonal of the square formed by the centres, and $\cos^2 EDF = \frac{2r^2}{4r^2} = \frac{1}{2}$, r being the radius of a shot. Hence the inclination of an edge to the base $= 45^0$.

Now $$\sin BC = \tan AB \cot ACB,$$

$$\text{or} \quad \tan I = \sqrt{2},$$

I being the inclination of a face of the box to the base.

Now $$r = ab \tan \frac{I}{2};$$

the second figure being a section of the pile made by a plane through the vertex and perpendicular to two opposite and parallel edges of the base;

$$\therefore\ ab = \frac{r}{\sqrt{2}} (\sqrt{3} + 1),$$

and side of base $= 6r + r\sqrt{2}\,(\sqrt{3} + 1)$.

$$\text{Altitude} = \frac{6r + r\sqrt{2}\,(\sqrt{3} + 1)}{2} \tan I$$

$$= \frac{6r + r\sqrt{2}\,(\sqrt{3} + 1)}{\sqrt{2}}$$

$\therefore$ volume of the enveloping pyramid

$$\frac{r^3}{3\sqrt{2}} (6 + \sqrt{6} + \sqrt{2})^3.$$

Problem XVI.

If α, β, γ be the angles which three diameters of a sphere make with each other, $\sigma = \dfrac{\alpha + \beta + \gamma}{2}$, *and* (a) *be the radius of the sphere, then the volume of the parallelopiped formed by planes which touch the sphere at the extremities of the three diameters*

$$\frac{4a^3}{\sqrt{\sin\sigma \sin(\sigma - \alpha) \sin(\sigma - \beta) \sin(\sigma - \gamma)}}$$

Fig. 12. Let Oa, Ob be two of the radii of the sphere whose centre is O. Then a plane passing through these will be perpendicular to the tangent planes at the points a, b; and therefore to their common intersection. Hence if ac, bc be perpendicular to Oa, Ob respectively, they will be drawn in these planes perpendicular to their common intersection; hence the angles α, β, γ are supplementary to the inclinations of the planes.

Let OA, OB, OC be the three edges, CT perpendicular to the plane of OA, OB; FHG a spherical triangle. Then if $FK = \delta$

$$\sin G = \frac{\sin\delta}{\sin FG},$$

$$\text{and} \quad \sin\delta = \sin G \cdot \frac{2}{\sin F \sin G}$$

$$\times \sqrt{-\cos S \cos(S - F) \cos(S - G) \cos(S - H)},$$

$$\text{where } 2S = F + G + H,$$

and the angles of this triangle are the inclinations of the planes, or equal $\pi - \alpha$, $\pi - \beta$, $\pi - \gamma$:

$$\therefore \sin\delta = \frac{2\sqrt{\sin\sigma \sin(\sigma - \alpha) \sin(\sigma - \beta) \sin(\sigma - \gamma)}}{\sin F},$$

$$\text{and} \quad 2a = OC \sin\delta,$$

$$\text{or } OC \quad \frac{a \sin F}{\sqrt{\sin\sigma \sin(\sigma - \alpha) \sin(\sigma - \beta) \sin(\sigma - \gamma)}}$$

$$\text{So } OB = \frac{a \sin G}{\sqrt{\sin\sigma \sin(\sigma - \alpha) \sin(\sigma - \beta) \sin(\sigma - \gamma)}},$$

and volume of the parallelopiped

$= \text{area } COB \times$ perpendicular distance between the planes

$= OC \,.\, OB \,.\, \sin FG \,.\, 2a$

$$= \frac{2a^3 \sin F \sin G}{\sin\sigma \sin(\sigma - \alpha) \sin(\sigma - \beta) \sin(\sigma - \gamma)} \times \frac{2}{\sin F \sin G} \sqrt{\sin\sigma \sin(\sigma - \alpha) \ldots}$$

$$\frac{4a^3}{\sqrt{\sin\sigma \sin(\sigma - \alpha) \sin(\sigma - \beta) \sin(\sigma - \gamma)}}.$$

Problem XVII.

If tangent planes be drawn at the extremities of any chord to a surface of the second order, their intersection lies in the plane diametral to that chord; and if the chord be supposed always to pass through a given point, the locus of the intersections of the pairs of tangent planes will be a plane; also when the given point is external, this locus is the plane of contact of all tangent planes which pass through the point.

Let the middle point of the chord be the origin; $x = mz$, $y = nz$ its equations; and let the equation to the surface be

$$ax^2 + by^2 + cz^2 + 2a'yz + 2b'xz + 2c'xy + 2a''x + 2b''y + 2c''z + d = 0,$$

under the condition $ma'' + nb'' + c'' = 0$.

Then the equation to the tangent plane is

$$(x'-x).(ax+b'z+c'y+a'')+(y'-y).(by+a'z+c'x+b'')$$
$$+(z'-z).(cz+a'y+b'x+c'')=0,$$

or since it is applied at the extremity of the chord $x'=mz'$, $y'=nz'$, its equation will be

$$z\{(ma+nc'+b')x'+(nb+mc'+a')y'+(na'+mb'+c)z'$$
$$+ma''+nb''+c''\}+x'a''+y'b''+z'c''+d=0,$$

or $(ma+nc'+b')x'+(nb+mc'+a')y'+(na'+mb'+c)z'$

$$+ma''+nb''+c''+\frac{x'a''+y'b''+z'c''+d}{z}=0,$$

and for the tangent plane at the other extremity,

$$(ma+nc'+b')x'+(nb+mc'+a')y'+(na'+mb'+c)z'$$
$$+ma''+nb''+c''-\frac{x'a''+y'b''+z'c''+d}{z}=0,$$

and adding these equations, their intersection will be in the plane

$$(ma+nc'+b')x'+(nb+mc'+a')y'+(na'+mb'+c)z'=0$$
$$(\text{since } ma''+nb''+c''=0),$$

that is, in the plane diametral to the chord.

II. Let the fixed point be the origin, $x=mz$, $y=nz$ the equations to a chord: by properly assuming the directions of the co-ordinate axes, the equation to surface may be made to take the form

$$ax^2+by^2+cz^2+2a'x+2b'y+2c'z+d=0.$$

The equation to a tangent plane applied at an extremity of the chord whose co-ordinates are x_1, y_1, z_1, will be

$$(ax_1+a')x'+(by_1+b')y'+(cz_1+c')z'+a'x_1+b'y_1+c'z_1+d=0,$$

or $(am+bn+c)x'+a'm+b'n+c'+\dfrac{a'x'+b'y'+c'z'+d}{z_1}=0\ldots(1).$

The equation to a tangent plane at the other extremity, whose co-ordinates are $-x_2$, $-y_2$, $-z_2$, is

$$(a'-ax_2)x'+(b'-by_2)y'+(c'-cz_2)z'-a'x_1-b'y_1-c'z_1+d=0,$$

or

$$-(am+bn+c)x'-(a'm+b'n+c')+\frac{a'x'+b'y'+c'z'+d}{z_2}=0\ldots(2),$$

and adding the equations (1) and (2), we have for the locus of the intersections of pairs of tangent planes, the plane whose equation is

$$a'x'+b'y'+c'z'+d=0.$$

III. The equation to the tangent plane at the point (x, y, z), supposing the fixed external point the origin, is

$$\left.\begin{array}{r}(ax+a')x'+(by'+b')y'+(cz+c')z'\\ +a'x+b'y+c'z+d\end{array}\right\}=0,$$

and since this passes through the origin, it is satisfied by $x'=0$, $y'=0$, $z'=0$;

$$\therefore\ a'x+b'y+c'z+d=0$$

is the relation between x, y, z for different positions of the tangent plane always passing through a fixed point, or it is the plane of contact of all these tangent planes, and coincides with that deduced in the previous part of this Problem.

Problem XVIII.

Prove that the stereographic projection of the sphere gives a representation which is similar to the part represented, immediately about any assigned point, but that the scale (i. e. *the ratio of any small line on the sphere to its projection*) *varies from point to point, and is* $= \frac{1}{2} \sec^2 \frac{\theta}{2}$, *where* θ *is the angle subtended at the centre of the sphere, by the arc of a great circle drawn from the part represented to that pole of the primitive plane which is opposite to the eye.*

Also shew that if (ϕ) *be the angle made by a plane through each point to be represented, and the axis, with a fixed plane through the axis, and if upon any plane a series of points be taken whose rectangular co-ordinates are*

$$x = a \left(\tan \frac{\theta}{2}\right)^{\lambda} \cos \lambda\phi, \quad y = a \left(\tan \frac{\theta}{2}\right)^{\lambda} \sin \lambda\phi,$$

where a, λ *are any constants, a projection will be made possessing the above property, the scale being now*

$$= \frac{a\lambda}{\mathrm{a}} \frac{\left(\tan \frac{\theta}{2}\right)^{\lambda}}{\sin \theta},$$

where a *is the radius of the sphere.*

Fig. 13. Let O be the centre of the sphere, PQ an elementary line on the sphere, the plane xy the plane of projection, A, E the poles of this plane;

$$AOP = \theta, \quad ROB = \phi, \quad OM = \rho.$$

Then $$\rho = a \tan \frac{\theta}{2}, \quad \delta\rho = \frac{1}{2} a \sec^2 \frac{\theta}{2} \delta\theta,$$

and elementary area of projection $= \delta\rho \,.\, \rho\delta\phi$

$$= \frac{a^2}{2} \frac{\sin\dfrac{\theta}{2}}{\cos^3\dfrac{\theta}{2}} \delta\theta\,\delta\phi.$$

Elementary area of sphere $= a^2 \sin\theta\,\delta\theta\,\delta\phi$;

$$\therefore \text{ ratio of these elements} = \frac{1}{2} \frac{\sin\dfrac{\theta}{2}}{\sin\theta\cos^3\dfrac{\theta}{2}}$$

$$= \left(\frac{1}{2}\sec^2\frac{\theta}{2}\right)^2.$$

Again, ratio of small lines

$$= \frac{\rho\,\delta\phi}{a\sin\theta\,\delta\phi} = \frac{a\tan\dfrac{\theta}{2}}{a\sin\theta}$$

$$= \frac{1}{2}\sec^2\frac{\theta}{2}.$$

Hence the ratio of elementary areas of the projection and the sphere, is the square of that of homologous elementary lines, and therefore the projection is similar to the part represented.

II. Again,

$$\left.\begin{aligned} d_\theta x &= a\lambda \,.\, \tfrac{1}{2}\sec^2\frac{\theta}{2} \,.\, \left(\tan\frac{\theta}{2}\right)^{\lambda-1} \cos\lambda\phi \\ d_\theta y &= a\lambda \,.\, \tfrac{1}{2}\sec^2\frac{\theta}{2} \,.\, \left(\tan\frac{\theta}{2}\right)^{\lambda-1} \sin\lambda\phi \end{aligned}\right\};$$

$\therefore$ if δs be an elementary line of the projection,

$$\left(\frac{\delta s}{\delta \theta}\right)^2 = \frac{a^2 \lambda^2 \left(\tan \frac{\theta}{2}\right)^{2\lambda}}{4 \sin^2 \frac{\theta}{2} \cos^2 \frac{\theta}{2}};$$

and scale of relation $$\frac{\delta s}{a \delta \theta} = \frac{a\lambda \left(\tan \frac{\theta}{2}\right)^{\lambda}}{a \sin \theta}.$$

We arrive at the same result by supposing (ϕ) to vary.

Also $$x^2 + y^2 = \rho^2 = a^2 \left(\tan \frac{\theta}{2}\right)^{2\lambda},$$

$$\frac{y}{x} = \tan \lambda \phi;$$

$$\therefore \text{ if } \frac{y}{x} = \tan \theta', \quad \theta' = \lambda \phi,$$

and $$\delta \rho = \frac{a\lambda}{2} \left(\tan \frac{\theta}{2}\right)^{\lambda - 1} \sec^2 \frac{\theta}{2} = \frac{a\lambda \left(\tan \frac{\theta}{2}\right)^{\lambda}}{\sin \theta} \delta \theta,$$

and elementary area of the projection

$$= \delta \rho \cdot \rho \delta \theta'$$

$$= \lambda \delta \phi . \frac{a\lambda}{\sin \theta} \left(\tan \frac{\theta}{2}\right)^{\lambda} \delta \theta . a \left(\tan \frac{\theta}{2}\right)^{\lambda}$$

$$= \frac{\left\{a\lambda \left(\tan \frac{\theta}{2}\right)^{\lambda}\right\}^2}{\sin \theta} \delta \theta \delta \phi,$$

and elementary area of sphere

$$= a \delta \theta, a \delta \phi \sin \theta;$$

∴ ratio of these small areas

$$= \frac{\left\{a\lambda\left(\tan\frac{\theta}{2}\right)^{\lambda}\right\}^{2}}{(a\sin\theta)^{2}},$$

and this equals the square of the scale, so that this possesses the same property as the last projection.

A full discussion of Problems 19 and 20 may be found in the *Théorie Analytique du Système du Monde*, par M. G. De Poutécoulant, Liv. v. Chap. 5, and Liv. iv. Chap. 5.

Then $$1 = A(x-a_1)(x-a_2)\dots(x-a_n)$$
$$+ Bx(x-a_2).(x-a_3)\dots(x-a_n)$$
$$+ \dots\dots\dots\dots\dots\dots\dots\dots$$
$$+ Lx(x-a_1)(x-a_2)\dots(x-a_{n-1});$$
$$\therefore A + B + C + \dots\dots + L = 0.$$

Now let $x = 0$.

Then $$1 = A.(-1)^n.a_1a_2\dots a_n,$$

and $$B + C + \dots\dots + L = -A$$
$$= \frac{(-1)^{n-1}}{x_1a_2\dots a_n}.$$

Now if $\frac{P}{Q}$ be the above fraction,

$$B = \frac{P_{x=a_1}}{d_{x=a_1}Q}, \quad C = \frac{P_{x=a_2}}{d_{x=a_2}Q}, \text{ \&c.} = \text{\&c.};$$

$$\therefore \frac{a_1^{-1}}{(a_1-a_2)(a_1-a_3)\dots(a_1-a_n)} + \frac{a_2^{-1}}{(a_2-a_1)(a_2-a_3)\dots(a_2-a_n)} \dots$$

or $$(-1)^{n-1}S = B + C + \dots\dots + L$$
$$= \frac{(-1)^{n-1}}{a_1a_2\dots a_n};$$
$$\therefore S = \frac{1}{a_1a_2\dots a_n}.$$

Again, let

$$\frac{x^m}{(x-a_1)(x-a_2)\dots(x-a_n)} = \frac{A}{x-a_1} + \frac{B}{x-a_2} + \dots + \frac{L}{x-a_n}.$$

Then $$x^m = A(x-a_2).(x-a_3)\ldots(x-a_n)$$
$$+ B(x-a_1)(x-a_3)\ldots(x-a_n)$$
$$+ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$+ L(x-a_1).(x-a_2)\ldots(x-a_{n-1}).$$

Now so long as m lies between 0 and $n-1$, we have

$$0 = A + B + C + \ldots\ldots + L$$

$$= \frac{a_1^m}{(a_1-a_2)\ldots(a_1-a_n)} + \frac{a_2^m}{(a_2-a_1)(a_2-a_3)\ldots(a_2-a_n)}$$
$$+ \;\&c.;$$

or $$S = 0.$$

But if $m = n-1$, then equating the coefficients of x^{n-1} on both sides of the above equation, we have

$$1 = A + B + C + \ldots\ldots + L.$$

Now A $$\frac{P_{x=a_1}}{d_{x=a_1}Q} = \frac{a_1^m}{(a_1-a_2)(a_1-a_3)\ldots(a_1-a_n)}$$

$$= \frac{\qquad}{(a_2-a_1)(a_3-a_1)\ldots(a_n-a_1)(-1)^{n-1}},$$

$$\therefore\; S = (-1)^{n-1}(A + B + \ldots\ldots + L)$$
$$= (-1)^{n-1}.$$

Problem II.

If PT, QT *be two tangents at the points* P *and* Q *of a parabola, whose focus is* S, *then* SP . SQ = ST², *and if* SP, SQ *include a given angle* α, *the locus of* T *will be a hyperbola whose eccentricity* = *sec* $\frac{\alpha}{2}$.

Let $\left.\begin{matrix} x_1, & y_1 \\ x_2, & y_2 \end{matrix}\right\}$ be the co-ordinates of $\left\{\begin{matrix} P \\ Q \end{matrix}\right.$,

$y^2 = 4ax$ the equation to the parabola.

The co-ordinates of T are found, by combining the equations $yy_1 = 2a(x + x_1)$, $yy_2 = 2a(x + x_2)$, to be

$$x = \sqrt{x_1 x_2}, \qquad y = \sqrt{a}\,(\sqrt{x_1} + \sqrt{x_2});$$

$$\begin{aligned} \therefore\ ST^2 &= (a - \sqrt{x_1 x_2})^2 + a(\sqrt{x_1} + \sqrt{x_2})^2 \\ &= a^2 + x_1 x_2 + a x_1 + a x_2 \\ &= (a + x_1) . (a + x_2) \\ &= SP . SQ. \end{aligned}$$

Again, x, y being the co-ordinates of T, we have

$$\tan\alpha = \frac{\dfrac{y_2}{a - x_2} - \dfrac{y_1}{a - x_1}}{1 + \dfrac{y_1 y_2}{(a - x_1) . (a - x_2)}}$$

$$= \frac{2\sqrt{a}\,(a - \sqrt{x_1 x_2}) . (\sqrt{x_2} - \sqrt{x_1})}{(a - x_1) . (a - x_2) + 4a\sqrt{x_1 x_2}} \ldots\ldots (1).$$

Also $\qquad x = \sqrt{x_1 x_2} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2),$

$\qquad\qquad y = \sqrt{a}\,(\sqrt{x_1} + \sqrt{x_2}) \ldots\ldots\ldots\ldots\ldots (3),$

and between these three equations we have to eliminate x_1, x_2.

Now, $$\left.\begin{aligned} 2\sqrt{x_1 a} &= y + \sqrt{y^2 - 4ax} \\ \text{and } 2\sqrt{x_2 a} &= y - \sqrt{y^2 - 4ax} \end{aligned}\right\};$$

$$\left.\begin{aligned} \therefore\ 4(x_1 + x_2) &= \frac{4}{a}(y^2 - 2ax) \\ \text{and } \sqrt{x_2} - \sqrt{x_1} &= -\frac{\sqrt{y^2 - 4ax}}{\sqrt{a}} \end{aligned}\right\}$$

Therefore from (1),

$$\{a^2 - (y^2 - 2ax) + x^2 + 4ax\}\tan\alpha = 2(a + x)\sqrt{y^2 - 4ax};$$

$$\therefore\ y^2 - 4ax + 2(a + x)\cot\alpha\sqrt{y^2 - 4ax} + \{(a + x)\cot\alpha\}^2$$
$$= (a + x)^2 \operatorname{cosec}^2\alpha;$$

$$\therefore\ \sqrt{y^2 - 4ax} + (a + x)\cot\alpha = \frac{a + x}{\sec\alpha},$$

$$\sqrt{y^2 - 4ax} = (a + x)\tan\frac{\alpha}{2},$$

$$y^2 - x^2\tan^2\frac{\alpha}{2} - \left(4a + 2a\tan^2\frac{\alpha}{2}\right)x = a^2\tan^2\frac{\alpha}{2}.$$

This is the equation to an hyperbola whose (semiaxes)^2 are $a'^2 = a^2$ and $b'^2 = a^2\tan^2\frac{\alpha}{2}$,

$$\text{and whose (eccentricity)}^2 = 1 + \frac{b'^2}{a'^2} = 1 + \tan^2\frac{\alpha}{2}$$
$$= \sec^2\frac{\alpha}{2};$$

$$\therefore\ \text{eccentricity} = \sec\frac{\alpha}{2}.$$

Problem III.

Shew that

$$(\cos\theta)^m + \left\{\cos\left(\frac{2\pi}{n}+\theta\right)\right\}^m + \left\{\cos\left(\frac{4\pi}{n}+\theta\right)\right\}^m + \dots$$
$$+ \left\{\cos\left(\frac{2(n-1)\pi}{n}+\theta\right)\right\}^m$$

is independent of θ *when* m *is less than* n, *but if* r *be the greatest integer contained in* $\frac{m}{n}$, *the above series can be reduced to the form*

$$A_0 + A_1 \cos n\theta + A_2 \cos 2n\theta + \dots + A_r \cos rn\theta,$$

where A_0, $A_1 \dots A_r$ *are independent of* θ.

We have by trigonometry,

$$2^{m-1}(\cos\theta)^m = \cos m\theta + a\cos(m-2)\theta + b\cos(m-4)\theta + \&c.$$

$$2^{m-1}\left\{\cos\left(\frac{2\pi}{n}+\theta\right)\right\}^m$$
$$= \cos m\left(\frac{2\pi}{n}+\theta\right) + a\cos\left\{(m-2)\cdot\left(\frac{2\pi}{n}+\theta\right)\right\} + \&c.$$

&c. = &c.

and each of these series terminates with a constant term when m is even.

$$\text{Also}\quad \cos\left\{\frac{2(n-1)}{n}\pi+\theta\right\} = \left\{\cos\left(\frac{2\pi}{n}-\theta\right)\right\};$$

therefore adding up these series we get, if S be the sum of the given series,

$$2^{m-1}S = \cos m\theta + a\cos(m-2)\theta + b\cos(m-4)\theta + \&c.$$
$$+ \left\{\cos m\left(\frac{2\pi}{n}+\theta\right) + \cos m\left(\frac{2\pi}{n}-\theta\right)\right\}$$

$$+ a\left[\cos\left\{(m-2).\left(\frac{2\pi}{n}+\theta\right)\right\}+\cos\left\{(m-2).\left(\frac{2\pi}{n}-\theta\right)\right\}\right]+\&c.$$

$$+\left\{\cos m\left(\frac{4\pi}{n}+\theta\right)+\cos m\left(\frac{4\pi}{n}-\theta\right)\right\}$$

$$+ a\left[\cos\left\{(m-2).\left(\frac{4\pi}{n}+\theta\right)\right\}+\cos\left\{(m-2).\left(\frac{4\pi}{n}-\theta\right)\right\}\right]+\&c.$$

+ constant term when m is even,

$$= \cos m\theta + a\cos(m-2)\theta + b\cos(m-4)\theta + \ldots$$

$$+ 2\cos m.\frac{2\pi}{n}\cos m\theta + 2a\cos(m-2)\frac{2\pi}{n}\cos(m-2)\theta + \ldots$$

$$+ 2\cos m.\frac{4\pi}{n}\cos m\theta + 2a\cos(m-2)\frac{4\pi}{n}\cos(m-2)\theta + \ldots$$

+ &c.

$$= \cos m\theta + a\cos(m-2)\theta + b\cos(m-4)\theta + \ldots$$

$$+ 2\cos m\theta\left\{\cos m\frac{2\pi}{n} + \cos m\frac{4\pi}{n} + \cos m\frac{6\pi}{n} + \ldots\right\}$$

$$+ 2a\cos(m-2)\theta\left\{\cos(m-2)\frac{2\pi}{n}\right.$$

$$\left. + \cos(m-2)\frac{4\pi}{n} + \cos(m-2)\frac{6\pi}{n} + \ldots\right\}$$

+ $(-\cos\theta)^m$ when n is even.

But when $m > n$, this series besides the above terms will contain some of the additional terms

$$h\cos n\theta + \ldots + k\cos 2n\theta + \ldots + p\cos rn\theta$$

$$+ 2h\cos n\theta\left\{\cos n.\frac{2\pi}{n} + \cos n.\frac{4\pi}{n} + \ldots\right\}$$

+ ....

$$+ 2k \cos 2n\theta \left\{\cos\left(2n \cdot \frac{2\pi}{n}\right) + \cos\left(2n \cdot \frac{4\pi}{n}\right) + \ldots\right\}$$

$$+ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$+ 2p \cos rn\theta \left\{\cos rn \cdot \frac{2\pi}{n} + \cos\left(rn \cdot \frac{4\pi}{n}\right) + \ldots\right\},$$

as often as the series m, $m-2$, $m-4$... contains multiples of n.

Now whenever m is not a multiple of n, the series

$$\cos m \cdot \frac{2\pi}{n} + \cos m \cdot \frac{4\pi}{n} + \ldots + \cos m \cdot \frac{(n-1)\pi}{n} = -\frac{1}{2},$$

and (n even) $= 0$ or -1 as m is odd or even, and when m is a multiple of n, its value depends on m and n.

Hence we collect that when $m < n$,

$$2^{n-1}S = \cos m\theta + a\cos(m-2)\theta + b\cos(m-4)\theta + \&c.$$

$$- \cos m\theta - a\cos(m-2)\theta - b\cos(m-4)\theta - \&c.$$

$$+ \text{a constant term when } m \text{ is even.}$$

Hence S is independent of θ in this case. But when $m > n$, there are, besides those that vanish or are constant, an additional set of terms, viz.

$$h\cos n\theta + k\cos 2n\theta + \ldots + p\cos rn\theta$$

$$+ h_1 \cos n\theta + k_1 \cos 2n\theta + \ldots + p_1 \cos rn\theta,$$

and hence S may be expressed in the form

$$A_0 + A_1 \cos n\theta + A_2 \cos 2n\theta + \ldots + A_r \cos rn\theta.$$

This may also be solved in the following manner: we have in Trigonometry the equation

$$(\cos\theta)^n + p_1(\cos\theta)^{n-1} + p_2(\cos\theta)^{n-2} + \ldots\ldots - \cos n\theta = 0.$$

Now since $\cos n\theta = \cos(2r\pi + n\theta)$, the n roots of this equation are

$$\cos\theta, \cos\left(\frac{2\pi}{n} + \theta\right), \cos\left(\frac{4\pi}{n} + \theta\right), \ldots \cos\left\{\frac{2(n-1)}{n}\pi + \theta\right\}.$$

Now the sums of the m^{th} powers of the roots of the above equation, that is, the sum of the series

$$(\cos\theta)^m + \left\{\cos\left(\frac{2\pi}{n}+\theta\right)\right\}^m + \ldots\ldots + \left\{\cos\left(\frac{2(n-1)}{n}\pi+\theta\right)\right\}^m$$

may be expressed in terms of the coefficients alone, excepting the last, so long as $m < n$. (Hymers' *Theory of Equations*, Art. 151.)

But if $m > n$, and if S_m be taken to represent the sum of the m^{th} powers of the roots of the above equation, we have

$$S_m + p_1 S_{m-1} + p_2 S_{m-2} + \ldots\ldots + p_n S_{m-n} = 0.$$

Hence

$$S_n + p_1 S_{n-1} + p_2 S_{n-2} + \ldots\ldots + n p_n = 0,$$

and S_n will involve p_n or $\cos n\theta$ besides constant terms.

Similarly

$$S_{2n} + p_1 S_{2n-1} + \ldots\ldots + p_n S_n = 0,$$

$$S_{rn} + p_1 S_{rn-1} + \ldots\ldots + p_n S_{(r-1)n} = 0.$$

Hence

S_{2n} will involve $p_n S_n$ or $\cos^2 n\theta$ or $\cos 2n\theta$, &c.

S_{3n} will involve $p_n S_{2n}$ or $\cos^3 n\theta$ or $\cos 3n\theta$, &c.

and S_{rn} .... $p_n S_{(r-1)n}$ or $\cos^r(n\theta)$ or $\cos rn\theta$, &c., together with constant terms.

Now the series expressing the value of S_m in terms of $S_{m-1}\, S_{m-2}$, &c. amongst others will contain $S_n\, S_{2n} \ldots S_{rn}$ as often as $m-1$, $m-2$ ... become multiples of n. Hence the whole value of S_m may in this case be expressed by

$$A_0 + A_1 \cos n\theta + A_2 \cos 2n\theta + \ldots + A_r \cos rn\theta.$$

Problem IV.

The foci of all those elliptical sections of a right cone, whose vertical angle is α, *which have the same eccentricity* e, *will lie in two conical surfaces, the tangent of the sum of whose semi-vertical angles* $= 2e \tan\frac{\alpha}{2}$.

Prove this, and find the vertical angle of each cone.

Fig. 14. Let V be the vertex of the cone; AB the major axis of an elliptical section, whose foci are S, H, and semiaxes a, β; draw HM perpendicular to VM.

Let $VAB = \theta$, $VSA = \phi$, $VA = a$, $VB = b$.

$$\text{Then} \quad \tan SVH = \frac{SH \sin\phi}{SV + SH\cos\phi}$$

$$= \frac{SH \sin^2\phi}{a\sin\theta + SH\sin\phi\cos\phi},$$

$$\text{and} \quad \sin\phi = \frac{a\sin(\phi+\theta)}{\alpha(1-e)};$$

$$\therefore \sin\phi = \frac{a\sin\theta}{\sqrt{A}}$$

$$\text{and} \quad \cos\phi = \frac{\alpha(1-e) - a\cos\theta}{\sqrt{A}}.$$

$$\text{where } A = 2a\alpha(1-e)\cos\theta + a^2(1-e)^2;$$

$$\therefore \tan SVH = \frac{SH \,.\, a^2\sin^2\theta}{aA\sin\theta + SH \,.\, a\sin\theta\,\{\alpha(1-e) - a\cos\theta\}}$$

$$= \frac{2a\alpha e\sin\theta}{a^2 - 2a\alpha\cos\theta + \alpha^2 - \alpha^2e^2}$$

$$= \frac{2a\alpha e\sin\theta}{a^2 - 2a\alpha\cos\theta + ab\sin^2\frac{\alpha}{2}}$$

Now $\quad \sin\theta = \sin\alpha \cdot \frac{b}{2a};$

$$\therefore\ a^2 a^2 \cos^2\theta = a^2 a^2 - a^2 b^2 \sin^2\alpha$$

$$= a^2\left(a^2 - \frac{b^2\sin^2\alpha}{4}\right)$$

$$= \frac{a^2}{4}(a - b\cos\alpha)^2,$$

since $\qquad 4a^2 = a^2 + b^2 - 2ab\cos\alpha;$

$$\therefore\ 2aa\cos\theta = a\,.\,(a - b\cos\alpha);$$

$$\therefore\ \tan SVH = e\,.\,\frac{ab\sin\alpha}{ab\cos\alpha + ab\sin^2\frac{\alpha}{2}}$$

$$\frac{\sin\alpha}{1 - 2\sin^2\frac{\alpha}{2} + \sin^2\frac{\alpha}{2}}$$

$$= 2e\,.\,\frac{\sin\frac{\alpha}{2}\cos\frac{\alpha}{2}}{\cos^2\frac{\alpha}{2}}$$

$$2e\tan\frac{\alpha}{2},$$

and SVH is the sum of the semi-vertical angles.

II. Again $e = \dfrac{AD}{AB} = \dfrac{\cos\left(\frac{\alpha}{2} + \theta\right)}{\cos\frac{\alpha}{2}}$ $\qquad$ (1)

Let $K = SVO$ = semi-vertical angle of one of the cones.

Then from the triangles AVS, BVS we have

$$\frac{1+e}{1-e} = \frac{\sin\left(\frac{\alpha}{2}+K\right)}{\sin\left(\frac{\alpha}{2}-K\right)} \cdot \frac{\sin\theta}{\sin(\alpha+\theta)};$$

$$\therefore \frac{\sin\left(\frac{\alpha}{2}+K\right)+\sin\left(\frac{\alpha}{2}-K\right)}{\sin\left(\frac{\alpha}{2}+K\right)-\sin\left(\frac{\alpha}{2}-K\right)}$$

$$= \frac{\sin(\alpha+\theta)+\sin\theta+e\,\{\sin(\alpha+\theta)-\sin\theta\}}{\sin(\alpha+\theta)-\sin\theta+e\,\{\sin(\alpha+\theta)+\sin\theta\}};$$

$$\therefore \tan\frac{\alpha}{2}\cot K = \frac{\tan\left(\frac{\alpha}{2}+\theta\right)+e\tan\frac{\alpha}{2}}{\tan\frac{\alpha}{2}+e\tan\left(\frac{\alpha}{2}+\theta\right)},$$

$$\tan\left(\frac{\alpha}{2}+\theta\right) = \frac{\sqrt{1-e^2\cos^2\frac{\alpha}{2}}}{e\cos\frac{\alpha}{2}}$$

$$\therefore \tan\frac{\alpha}{2}\cot K = \frac{\sqrt{1-e^2\cos^2\frac{\alpha}{2}}+e^2\sin\frac{\alpha}{2}}{\sqrt{1-e^2\cos^2\frac{\alpha}{2}}+e\sin\frac{\alpha}{2}}$$

Similarly the vertical angle of the other cone may be found.

PROBLEM V.

Two dice are placed together so as to form a parallelopiped: determine the chance that two or more contiguous faces of the dice may have the same marks.

Each of the dice is so marked that the sum of the numbers on any two opposite faces = 7. And when any two particular faces of the dice are in contact, one of them may be turned round an axis in the direction of the length of the parallelopiped, and assume four different positions: hence the whole number of positions of the dice which give different combinations of the contiguous faces is 144.

Let A, B be the dice.

First, if A's ace be in contact with B's ace, there are four positions favourable to two contiguous faces* being the same.

If A's ace be combined with B's 2,
the No. of favourable positions = 1,

.... 3 .... = 1,

.... 4 .... = 1,

.... 5 .... = 1,

.... 6 .... = 2.

If A's ace be therefore presented successively to each of the faces of B, there are on the whole 10 favourable positions, and similarly with regard to the other 5 faces of A. Hence there are on the whole 60 different positions of the dice favourable to the chance of two or more being contiguous, which is therefore $\frac{60}{144}$ or $\frac{5}{12}$.

* Those faces which are in contact are included.

Problem VI.

When two pith balls repelling one another with forces varying as $\frac{1}{(dist.)^2}$ *are connected by a fine thread passing over a fixed point A, and are also acted on by gravity, find their position of rest.*

Fig. 51. Since the tension t of the thread is the same throughout, the resultant of the tensions must bisect the angle PAQ and pass through the centre of gravity G of P and Q. Let c be the length of the thread, θ the angle which either portion AP or AQ makes with the vertical, $\frac{f}{r^2}$ the repulsive force.

Then by the triangle of forces,

$$\frac{t}{P} = \frac{AP}{AG}, \qquad \frac{Q}{t} = \frac{AG}{AQ};$$

$$\therefore \frac{Q}{P} = \frac{AP}{c - AP};$$

$$\therefore AP = \frac{cQ}{P+Q}, \qquad AQ = \frac{cP}{P+Q} \qquad .(1).$$

Again,

$$r^2 (P+Q)^2 = c^2 (P^2 + Q^2) - 2PQc^2 \cos 2\theta, \text{ from triangle } PAQ$$

$$= c^2 (P+Q)^2 - 4PQc^2 \cos^2\theta;$$

$$\therefore (c^2 - r^2) . (P+Q)^2 = 4PQc^2 \cos^2\theta \quad ....(2).$$

Also by the triangle of forces,

$$\frac{f}{P.r^2} = \frac{PG}{AG} = \frac{rQ}{\sqrt{PQ}\sqrt{c^2 - r^2}};$$

$$\therefore f^2 (c^2 - r^2) = PQ . r^6,$$

$$\text{or } r^6 = \frac{4f^2c^2\cos^2\theta}{(P+Q)^2}, \quad \text{from (2)},$$

$$r^2 = \left(\frac{2fc\cos\theta}{P+Q}\right)^{\frac{2}{3}},$$

and substituting in the above equation, we have

$$2f\cos\theta\,.\,(P+Q)^2 = c^2\left(P^2 - 2PQ\cos 2\theta + Q^2\right)^{\frac{3}{2}} \ldots\ldots (3),$$

and the equations (1) and (3) determine the position of equilibrium of each ball.

Problem VII.

Supposing the earth a homogeneous spheroid of equilibrium, the time of descent of a body let fall from any point P *on the surface down a hole bored to the centre* C, *varies as* CP, *and the velocity at the centre is constant.*

Fig. 15. Let P be the position of the body at the time t from the commencement of the motion, a', c' the semiaxes of the spheroid passing through P and similar to the spheroidal surface whose semiaxes are a, c. Let X, Y be the forces parallel to the major and minor axes which are taken for the axes of x and y.

Then since the force acts in the common normal to all the similar surfaces, we have

$$\frac{Y}{X} = \frac{a^2}{c^2}\,\frac{y}{x},$$

$$X = \mu x, \qquad Y = \mu\frac{a^2}{c^2}y,$$

where μ depends on a and c.

Let $\qquad CP = r, \qquad PCA = \theta.$

Then $\qquad d_t^2 r = -\mu\left(x\cos\theta + \frac{u}{c^2}\,y\sin\theta\right)$

$$-\mu \left\{x^2 + \frac{u}{c'^2}\,y^2\right\}$$

$$\frac{\mu a^1}{r}$$

Let R be the value of r at the surface.

Then $\qquad \frac{R^2}{r^2} = \left(\frac{\cos^2\theta}{a'^2} + \frac{\sin^2\theta}{c'^2}\right) \div \left(\frac{\cos^2\theta}{a^2} + \frac{\sin^2\theta}{c^2}\right)$

$$\therefore\ d_t^2 r = -\frac{\mu a^2}{R^2}\,.\,r;$$

$$\therefore\ (d_t r)^2 = C - \frac{\mu a^2}{R^2}\,.\,r^2,$$

and when $t = 0$, $r = R$ and $d_t r = 0$;

$$\therefore\ 0 = C - \frac{\mu a^2}{R^2}\,.\,R^2;$$

$$\therefore\ (d_t r)^2 = \frac{\mu a^2}{R^2}\,.\,(R^2 - r^2)$$

μa^2 at the centre, which is a constant quantity.

Again, $\qquad d_r t = \frac{R}{a}\sqrt{\frac{1}{\mu}}\,.\,\frac{-1}{\sqrt{R^2 - r^2}},$

taking the negative sign, since r diminishes as t increases

Hence $\qquad t = C + \frac{R}{a\sqrt{\mu}}\,.\cos^{-1}\frac{r}{R}.$

and when $t = 0$ $r = R$; $\therefore C = 0$,

$$\text{and } t = \frac{R}{a\sqrt{\mu}} \cos^{-1}\frac{r}{R},$$

$$\text{or time to centre} = \frac{R}{a} \cdot \frac{\pi}{2\sqrt{\mu}},$$

$$\propto CP.$$

Problem VIII.

When an oblique cone upon an elliptical base (whose semi-axes are α, β and centre C) admits of a circular section, if D be the foot of the perpendicular drawn from the vertex upon the base, and CD = a, the centres of all the circles will lie in two straight lines which meet the base in two points, the sum of the reciprocals of whose distances from C $= \dfrac{2a}{\alpha^2 - \beta^2}$.

Let the plane of the elliptic base be the plane of xy, the centre being the origin; and let the plane which passes through the axis of the cone and is perpendicular to the base, intersect the base in a diameter inclined at an angle ϕ to the major axis, which take for the axis of x. Let a, c be the co-ordinates of the vertex of the cone. Then the equations to a generating line are

$$x' - a = \frac{x - a}{z - c}(z' - c) \qquad y' = \frac{y}{z - c}(z' - c),$$

and where it meets the base

$$x' = \frac{az - cx}{z - c} \qquad y' = \frac{-cy}{z - c}.$$

Now the equation to the elliptic base is

$$\left(\frac{x'\cos\phi - y'\sin\phi}{\alpha}\right)^2 + \left(\frac{x'\sin\phi + y'\cos\phi}{\beta}\right)^2 \quad 1,$$

and as the co-ordinates found above satisfy this equation, the equation to the cone is,

$$\left\{\frac{(\alpha z - cx)\cos\phi + cy\sin\phi}{\alpha}\right\}^2 + \left\{\frac{(\alpha z - cx)\sin\phi - cy\cos\phi}{\beta}\right\}^2 = (z-c)^2.$$

Now if the cone admit of a principal plane and therefore of a circular section, which is always perpendicular to it, it is evident that it must be the plane of xz which is perpendicular to the plane of the base and passes through the axis. Let h, k, l, be the co-ordinates of one of the centres D: then taking D as the origin, we have (Fig. 16)

$$x = ON = OV + LT = x'\cos\theta + h,$$
$$y = NM = CL = y' + k,$$
$$z = PM = PT + CV = x'\sin\theta + l,$$

θ being the inclination of the plane of the circle to xy.

Hence the equation to the circle is

$$\left[\frac{\{(\alpha\sin\theta - c\cos\theta)x' + \alpha l - ch\}\cos\phi + c(y'+k)\sin\phi}{\alpha}\right]^2$$
$$\left[\frac{\{(\alpha\sin\theta - c\cos\theta)x' + \alpha l - ch\}\sin\phi - c(y'+k)\cos\phi}{\beta}\right]^2$$
$$(x'\sin\theta + l - c)^2.$$

Hence equating to zero the coefficient of $x'y'$, since the axes are rectangular, we have

$$\sin\phi\cos\phi\left(\frac{1}{\alpha^2} - \frac{1}{\beta^2}\right) = 0.$$

$$\text{Also } \{ck\sin\phi + (\alpha l - ch)\cos\phi\} \cdot c\sin\phi = \{ck\cos\phi + (\alpha l - ch)\sin\phi\} \cdot c\cos\phi.$$

Hence $\phi = 0$, or $\phi = \frac{\pi}{2}$; that is, the axis of x is either the major or minor axis; suppose the former, then $\phi = 0$ and then $k = 0$. Hence equating the coefficients of x'^2 and y'^2, and equating to zero the coefficient of x', we have

$$\left(\frac{a \sin\theta - c \cos\theta}{a}\right)^2 - \sin^2\theta = \frac{c^2}{\beta^2} \quad (1),$$

and $$\frac{al - ch}{a^2}(a \sin\theta - c\cos\theta) = (l - c)\sin\theta \quad (2).$$

Equation (2) shews that the locus of the centres are two straight lines, since there are two values of $\tan\theta$ derived from (1). Let $l = 0$, and let $h_1\, h_2$ be the two corresponding values of h, then

$$\frac{a^2}{h_1} = \frac{a \sin\theta - c\cos\theta}{\sin\theta}$$

$$= a - c \cot\theta.$$

So $$\frac{a^2}{h_2} = a - c\cot\theta_1;$$

$$\therefore\ a^2\left(\frac{1}{h_1} + \frac{1}{h_2}\right) = 2a - c(\cot\theta + \cot\theta_1).$$

Now $\cot\theta + \cot\theta_1 = \dfrac{2a\beta^2}{c(\beta^2 - a^2)}$ from (1);

$$\therefore\ a^2\left(\frac{1}{h_1} + \frac{1}{h_2}\right) = 2a - \frac{2a\beta^2}{\beta^2 - a^2} = \frac{2aa^2}{a^2 - \beta^2};$$

$$\frac{1}{h_1} + \frac{1}{h_2} = \frac{2a}{a^2 - \beta^2}.$$

Problem IX.

Two weights, P *and* Q, *are attached to a third weight* W *by means of two strings* PAW, QAW, *passing through a small ring* A, *below which* W *hangs freely; determine the nature of the curve on which* P *and* Q *will rest in all positions compatible with the conditions of the system, and deduce the particular case in which all the weights* P, Q *and* W *are equal.*

Fig. 17. Let BPQ be the curve; A the ring;

$AP = r$, $AQ = r_1$, $AM = x$, $MP = y$, $AW = z$,

$AN = x'$, $PAW = c$, $QAW = c_1$.

Then the conditions of the problem require that the distance of the centre of gravity of P, Q, W from a horizontal plane through A should be constant and $= h$;

$$\therefore \; Px + Qx' + Wz = (P + Q + W)\,h,$$

$$\text{and } r + z = c, \quad r_1 + z = c_1.$$

Let $x = f(r)$ be the equation to the curve.

$$\text{Then } x' = f(r_1) = f(c_1 - c + r) = f(a + r)$$

$$\text{where } a = c_1 - c.$$

Let $f(r) = u_r$. Then $f(a + r) = u_{r+a}$,

$$\text{and } Pu_r + Qu_{r+a} + W(c - r) = (P + Q + W)h.$$

Let $\qquad r = ay, \qquad u_r = v_y.$

Then $Pv_y + Qv_{y+1} + W(c - ay) = (P + Q + W)h$,

$$v_{y+1} + \frac{P}{Q}v_y - \frac{Wa}{Q}y = \frac{(P + Q + W)h - Wc}{Q}.$$

Let $\qquad v_y = w_y + k.$

Then $$w_{y+1} + \frac{}{Q} w_y = \frac{Wa}{Q} y,$$

$$\text{if } k = \frac{(P+Q+W)h - Wc}{P+Q}.$$

The solution of the above equation is (Hymers' *Finite Differences*, Art. 74.) if $-\dfrac{P}{Q} = \dfrac{1}{\beta}$,

$$w_y \beta^{y-1} = C + \frac{Wa}{Q} \Sigma(y\beta^y)$$

$$= C + \frac{Wa}{Q} \left\{ \frac{y\beta^y}{\beta - 1} - \frac{\beta^{y+1}}{(\beta - 1)^2} \right\}.$$

Here the constant may have any value which does not alter when y becomes $y+1$. Suppose, for simplicity, that $C = 0$.

Then $$w_y = \frac{Wa\beta}{Q(\beta - 1)^2} \cdot \{(\beta - 1)\, y - \beta\},$$

or $$x = v_y = w_y + k$$

$$= \frac{Wa\beta}{Q(\beta - 1)^2} \cdot \left\{(\beta - 1) \cdot \frac{r}{a} - \beta\right\} + k,$$

or $$\left\{ \frac{Q(\beta - 1)^2}{W\beta} \cdot (x - k) + a\beta \right\}^2 \qquad - 1)^2 \cdot (x^2 + y^2),$$

which is the equation to an hyperbola, its major axis being vertical. When $P = Q = W$, then $\beta = -1$, and

$$\{4(x - k) + a\}^2 = 4(x^2 + y^2).$$

Also if $a = 0$ or P, Q be united into one weight $= 2W$, and $k = 0$ or $c = 3h$, we shall have $y = x\sqrt{3}$, or we arrive at the case of an inclined plane, whose inclination to the horizon is 30^0, and the depth of the centre of gravity below the ring is one-third the length of the string, which shews that the string is parallel to the plane; and we know that there is equilibrium in this case in every position.

Obs. The statical principle employed in this problem may be deduced either from the equation of virtual velocities, which is $P\delta x + Q\delta x' - W\delta r - 0$, or (since $z + r = c$) $P\delta x + Qdx' + W\delta z = 0$, or from the equations of equilibrium. Let T, T_1, be the tensions of the strings: R, R' the reactions: θ, ϕ the inclinations of the normals to the vertical.

$$\text{Then} \quad R\cos\theta + T\frac{x}{r} = P, \quad R_1\cos\phi + T_1\frac{x_1}{r} = Q,$$

$$T\frac{y}{r} = R\sin\theta \; T_1\frac{y_1}{r_1} = R_1\sin\phi,$$

$$T + T_1 = W, \quad x^2 + y^2 = r^2.$$

Eliminate $R_1 R$ and we have $P = Td_x r \; Q = T_1 d_{x_1} r_1$;

$$\therefore \; Pd_r x + Qd_r x_1 = W \text{ (since } r - r_1 = c_2);$$

$$\therefore \; Px + Qx_1 + Wz = \text{constant}.$$

Problem X.

Three stars whose differences of right ascension and distances are known, have the same azimuth at each of two observations taken after a given interval; determine the latitude.

Fig. 17 *bis*.

Let $S'S'' = \gamma'$, $S'S = \gamma$, $PS' = \phi$, $S'PS'' = \alpha'$, $SPS' = \alpha$,

$$PZ = 90^0 - l, \quad PS'S = \theta.$$

$$\text{Then} \quad \left.\begin{aligned} \cot\gamma'\sin\phi &= \cos\phi\cos(\pi - \theta) + \sin\theta\cot\alpha' \\ \cot\gamma\sin\phi &= \cos\phi\cos\theta + \sin\theta\cot\alpha \end{aligned}\right\};$$

$$\therefore \; (\cot\gamma + \cot\gamma')\sin\phi = (\cot\alpha + \cot\alpha')\sin\theta,$$

$$\cot\gamma\sin\phi = \cos\phi\cos\theta + \cot\alpha \,.\, \frac{\cot\gamma + \cot\gamma'}{\cot\alpha + \cot\alpha'}\sin\phi,$$

$$\text{and} \quad \cos\phi\cos\theta = \sin\phi\left(\frac{\cot\gamma\cot\alpha' - \cot\alpha\cot\gamma'}{\cot\alpha + \cot\alpha'}\right),$$

$$\cot^2\phi\left\{1-\left(\frac{\cot\gamma+\cot\gamma'}{\cot a+\cot a'}\right)^2\sin^2\phi\right\}$$

$$=\sin^2\phi\left(\frac{\cot\gamma\cot a'-\cot a\cot\gamma'}{\cot a+\cot a'}\right)^2.$$

Whence ϕ is known, and $\sin PO=\sin\theta\sin\phi$

$$\left(\frac{\cot\gamma+\cot\gamma'}{\cot a+\cot a'}\right)\sin^2\phi,$$

and $\tan l=\cos h\cot PO$ is therefore known.

Problem XI.

Two radii SP, SQ *of a curve are drawn so as to include a given angle; find its nature when the inclination of the tangents is constant, and shew from the result that a parabola having* S *for its focus is a particular case.*

Fig. 18. Let PQ be the curve, $PSQ=a$ the given angle, ϕ, ψ the inclinations of the tangents at P, Q to the radii vectores, γ the angle between the tangents.

Then $$\phi-\psi=\pi-(a+\gamma);$$

$$\therefore\ -\tan(a+\gamma)=\frac{\tan\phi-\tan\psi}{1+\tan\phi\tan\psi}.$$

Let $\tan\psi=u_\theta$. Then $\tan\phi=u_{\theta+a}$.

$$\therefore\ u_{\theta+a}-u_\theta=-\tan(a+\gamma)-u_\theta u_{\theta+a}\tan(a+\gamma),$$

$$\text{or}\ u_\theta u_{\theta+a}+(u_{\theta+a}-u_\theta)\cot(a+\gamma)+1=0.$$

Let $\theta=ax$, $u_\theta=v_x$. Then $u_{\theta+a}=v_{x+1}$;

$$\therefore\ v_x v_{x+1}+(v_{x+1}-v_x)\cot(a+\gamma)+1=0.$$

The solution of which is (Herschell's *Examples*),

$$v_x=\tan\left\{-x(a+\gamma)+\tan^{-1}C\right\};$$

$$\therefore\ u_\theta = \tan\left(-\frac{a+\gamma}{a}\theta + \tan^{-1}C\right),$$

$$\frac{d_\theta r}{r} = \frac{\cos\left(-\frac{a+\gamma}{a}\theta + \tan^{-1}C\right)}{\sin\left(-\frac{a+\gamma}{a}\theta + \tan^{-1}C\right)}.$$

Now let $C = \tan\frac{\pi\theta}{a}$, which does not change when θ becomes $\theta + a$;

$$\frac{d_\theta r}{r} = \frac{\cos\frac{\pi-(a+\gamma)}{a}\theta}{\sin\frac{\pi-(a+\gamma)}{a}\theta},$$

$$\log_e\left(\frac{r}{a}\right) = \frac{-a}{a+\gamma-\pi}\log_e \sin\frac{\pi-(a+\gamma)}{a}\theta$$

$$= \frac{a}{a+\gamma-\pi}\log_e \frac{1}{\sin\frac{\pi-(a+\gamma)}{a}\theta}$$

$$\text{or}\quad \frac{r}{a} = \frac{1}{\left\{\sin\frac{\pi-(a+\gamma)}{a}\theta\right\}^{\frac{a}{a+\gamma-\pi}}},$$

the general solution.

Let $$a = \pi,\quad \gamma = \frac{\pi}{2}.$$

Then $$\frac{r}{a} = \frac{1}{\sin^2\frac{\theta}{2}} = \frac{2}{1-\cos\theta},$$

$$r = \frac{2a}{1-\cos\theta},$$

which is the equation to a parabola with focus S.

Problem XII.

A sphere rests upon a string fastened at its extremities to two fixed points; shew that if the arc of contact of the string and sphere be not less than $2\tan^{-1}\dfrac{48}{55}$, the sphere may be divided into two equal portions by means of a vertical plane without disturbing the equilibrium.

Fig. 19. Let $CA = a$, $DCB = 2\alpha$ the arc of contact.

Let $ACP = \theta$, p the pressure at P, W the weight of the sphere, G the centre of gravity of one hemisphere.

The tension of the string, and therefore the pressure on the sphere is the same at every point,.

$$\text{pressure on an element at } P = pa\delta\theta \text{ ultimately,}$$

$$\text{and vertical part} = pa\delta\theta\cos\theta;$$

$$\therefore \text{ whole vertical pressure} = 2pa\sin\alpha = W.$$

Again, the moment of pressure at P about A

$$= pa\delta\theta \,.\, a\sin\theta.$$

$$\text{Whole moment} = pa^2(1-\cos\theta) = 2pa^2\sin^2\frac{\theta}{2}\begin{pmatrix}\theta = 0\\ \theta = \alpha\end{pmatrix}$$

$$= 2pa^2\sin^2\frac{\alpha}{2};$$

$$\therefore\ 2pa^2\sin^2\frac{\alpha}{2} = \frac{W}{2}\,.\,CG = \frac{W}{2}\,.\,\frac{3a}{8},$$

if each hemisphere be in equilibrium;

$$\therefore\ 2a^2\sin^2\frac{\alpha}{2}\,.\,\frac{W}{2a\sin\alpha}\quad\frac{3Wa}{16};$$

$$\frac{\sin^2 \frac{\alpha}{2}}{\sin \alpha} = \frac{3}{16}, \quad \text{and } \tan \frac{\alpha}{2} = \frac{3}{8},$$

$$\text{and } \tan \alpha = \frac{\frac{3}{4}}{1 - \frac{9}{64}} = \frac{48}{55}$$

$$\therefore \; 2\alpha = 2 \tan^{-1} \left(\frac{48}{55}\right).$$

This gives the limit at which equilibrium will subsist; and hence, if the arc of contact be not less than $2 \tan^{-1} \left(\frac{48}{55}\right)$ there *must* be equilibrium.

Problem XIII.

When the centre of gravity is at a proper depth in a fluid whose density varies as the depth, an uniform equilateral triangle will rest in any position; and a right-angled triangle may be kept with either side horizontal, by a couple whose moment will be the same whichever side and whichever angle is uppermost.

Fig. 20. The plane of the triangle must be vertical, otherwise the centres of gravity of the solid and of the fluid displaced by it cannot lie in the same vertical line, which is a necessary condition of equilibrium.

Suppose then that the plane of the triangle ABC intersects the surface of the fluid in VD; AK which is perpendicular to BC meeting this line in V, and inclined at an angle α to it.

$AN = x$, $NP = y$ the co-ordinates of any point in the triangle, ρ the density of the fluid at the depth $HG = d$ of the centre of gravity, a a side of the triangle.

Then density at $P = \rho . \dfrac{VN \sin \alpha - y \cos \alpha}{d}$,

and $VN = x + \dfrac{d}{\sin \alpha} - \dfrac{a}{\sqrt{3}}$;

therefore whole mass of fluid displaced

$$= \frac{\rho}{d} \int_x \int_y \left(x \sin \alpha + d - \frac{a \sin \alpha}{\sqrt{3}} - \cos \alpha . y\right)$$

$$= \frac{\rho}{d} \int_x \left\{ \left(x \sin \alpha + d - \frac{a \sin \alpha}{\sqrt{3}} \right) y - \frac{\cos \alpha . y^2}{2} \right\}$$

between limits $y = \pm \dfrac{x}{\sqrt{3}}$;

therefore whole mass $= \dfrac{2\rho}{d\sqrt{3}} \int_x \left\{ x^2 \sin \alpha + \left(d - \dfrac{a \sin \alpha}{\sqrt{3}} \right) x \right\}$

between limits $x = 0$, $x = \dfrac{a\sqrt{3}}{2}$;

therefore whole mass of fluid displaced

$$= \frac{2\rho}{d\sqrt{3}} \left\{ \frac{\sin \alpha}{8} \sqrt{3} . a^3 + \left(d - \frac{a \sin \alpha}{\sqrt{3}} \right) \frac{3a^2}{8} \right\}$$

$$= \frac{\rho a^2 \sqrt{3}}{4} = \rho \times \text{area of triangle}$$

$= \text{mass of triangle.}$ (See Note 1.)

Hence if the centre of gravity be at such a depth that the density of the fluid equals that of the solid, then one condition of equilibrium will be satisfied, whatever be the value of α. We must next prove that the

centre of gravity of the fluid displaced lies in the vertical GH. For this purpose we have mass of an element of fluid at P

$$= \frac{\rho}{d}\delta x \delta y \left(x \sin a + d - \frac{a \sin a}{\sqrt{3}}\right) - \frac{\rho \cos a}{d} y \delta x \delta y,$$

and $ST = NG \cos a - y \sin a$

$$= \frac{a \cos a}{\sqrt{3}} - (x \cos a + y \sin a).$$

Moment of an element at P about HG

$$\frac{\rho}{d}\left\{\int_x \int_y (x \sin a - y \cos a) \frac{a \cos a}{\sqrt{3}}\right.$$

$$- \int_x \int_y (x \cos a + y \sin a) \cdot (x \sin a - y \cos a)$$

$$\left. - \left(d - \frac{a \sin a}{\sqrt{3}}\right) \int_x \int_y (x \cos a + y \sin a) + \int_x \int_y \left(d - \frac{a \sin a}{\sqrt{3}}\right) \frac{a \cos a}{\sqrt{3}}\right\}.$$

Now $\int_x \int_y (x \sin a - y \cos a) = \frac{1}{4} a^3 \sin a$, between the limits before specified,

$$\int_x \int_y (x \cos a + y \sin a) = \frac{1}{4} a^3 \cos a,$$

$$\int_x \int_y (x \cos a + y \sin a) \cdot (x \sin a - y \cos a)$$

$$= \int_x \left(x^2 y - \frac{y^3}{3}\right) \sin a \cos a + \frac{x y^2}{2} (\sin^2 a - \cos^2 a)$$

$$= 2 \sin a \cos a \cdot \frac{x^4}{4} \left(\frac{1}{\sqrt{3}} - \frac{1}{9\sqrt{3}}\right)$$

$$= \frac{a^4 \sin a \cos a}{4\sqrt{3}},$$

$$\frac{a \cos a}{\sqrt{3}} \left(d - \frac{a \sin a}{\sqrt{3}}\right) \int_x \int_y 1 = \frac{a^3 \cos a}{4} \left(d - \frac{a \sin a}{\sqrt{3}}\right);$$

$$\therefore \frac{w}{\rho} \times \text{moment of fluid about } HG$$

$$\frac{a \cos a}{\sqrt{3}} \cdot \frac{a^3 \sin a}{4} - \left(d - \frac{a \sin a}{\sqrt{3}}\right) \frac{a^3 \cos a}{}$$

$$\frac{a^4 \sin a \cos a}{4\sqrt{3}} + \frac{a^3 \cos a}{\cdot} \left(d - \frac{a \sin a}{\sqrt{3}}\right)$$

$$0.$$

Hence the centre of gravity of the fluid displaced lies in the same vertical with that of the solid, whatever be the angle a; and thus the second condition of equilibrium is satisfied, whatever be the position of the triangle, provided only that the centre of gravity of the triangle be at the depth before alluded to.

Again (fig. 21). Let ABC be a right-angled triangle, whose centre of gravity G is at a distance (d) from the surface of the fluid, and one side (AC) of which is horizontal.

Let $BM = x$, $MP = y$ be the co-ordinates of any element of the triangle, whose sides are a, b, c.

Let ρ be the density of the fluid at depth (d).

Then mass of an element $= \frac{\rho}{d}(d + \frac{2c}{3} - x)\,\delta x\,\delta y$, and whole mass of fluid displaced

$$\frac{\rho}{d} \int_x \int_y \left(d + \frac{2c}{3} - x\right)$$

$$\text{between limits} \quad \left.\begin{aligned} y &= 0, & y &= x \cdot \frac{b}{c} \\ x &= 0, & x &= c \end{aligned}\right\}$$

Hence mass of fluid displaced

$$= \frac{\rho}{d}\int_x \left\{\left(d + \frac{2c}{3}\right)\frac{bx}{c} - \frac{bx^2}{c}\right\}$$

$$= \frac{\rho}{d}\left\{\left(d + \frac{2c}{3}\right)\frac{bc}{2} - \frac{bc^2}{3}\right\}$$

$$= \frac{\rho}{d}\cdot\frac{bcd}{2} = \frac{bc\rho}{2}.$$

Hence if the centre of gravity of the triangle be placed at a depth where the density of the fluid is equal to its own, one condition of equilibrium will be satisfied.

Again, mass of an element of fluid

$$= \frac{\rho}{d}\left(d + \frac{2c}{3} - x\right)\delta x \delta y, \text{ ultimately,}$$

and moment of the whole fluid displaced

$$= \frac{\rho}{d}\int_x\int_y \left(d + \frac{2c}{3} - x\right).\left(\frac{b}{3} - y\right)$$

$$= \frac{\rho}{9d}\int_x (3d + 2c - 3x).\left(by - \frac{3y^2}{2}\right)$$

$$= \frac{\rho}{9d}\int_x (3d + 2c - 3x).\left(\frac{b^2x}{c} - \frac{3b^2x^2}{2c^2}\right),$$

(between limits $x = 0$, $x = c$).

Hence moment of the fluid displaced

$$= \frac{\rho}{9d}\left\{\frac{3b^2}{c}\left(\frac{3c^3}{8} - \frac{c^3}{3}\right)\right\}$$

$$= \frac{\rho}{9d}\cdot\frac{3b^2}{c}\cdot\frac{c^3}{24}$$

$$= \frac{b^2c^2\rho}{72d},$$

and this result shews that, provided (d) be a given value, the moment of the couple which is necessary to maintain the triangle in equilibrium is the same whichever side or whichever angle be uppermost.

PROBLEM XIV.

There will always be an ellipse and hyperbola, whose major axes are at right angles, which have a contact of the third order with a curve at any proposed point, and whose eccentricities e, e_1 *are connected by the equation*

$$\frac{1}{e_1^2} - \frac{1}{e^2} = 1.$$

Let h, k be the co-ordinates of the centre of the ellipse which has a contact of the third order with a curve at a given point; θ the inclination of its axis major to the axis of X.

Then if (X, Y) be the co-ordinates of any point; $X_1 Y_1$ those of the same point referred to the centre as origin and the principal axes as axes of co-ordinates, they are connected by the equations

$$\left.\begin{aligned} X_1 &= (X - h)\cos\theta + (Y - k)\sin\theta \\ Y_1 &= -(X - h)\sin\theta + (Y - k)\cos\theta \end{aligned}\right\},$$

and, if e be the eccentricity,

$$Y_1^2 = (1 - e^2)(a^2 - X_1^2),$$

or if x, y be the co-ordinates of the point of contact,

$$\{(y - k)\cos\theta - (x - h)\sin\theta\}^2 = (1 - e^2)$$
$$[a^2 - \{(x - h)\cos\theta + (y - k)\sin\theta\}^2].$$

Let $\qquad p = d_x y, \quad q = d_x^{\,2} y, \quad r = d_x^{\,3} y.$

Then

$$(p\cos\theta - \sin\theta)\left\{(y-k)\cos\theta - (x-h)\sin\theta\right\}$$
$$= -(1-e^2)\left\{(x-h)\cos\theta + (y-k)\sin\theta\right\}(\cos\theta + p\sin\theta),$$

or $\quad (x-h)\left\{1 - e^2\cos\theta\,(\cos\theta + p\sin\theta)\right\}$
$$= (y-k)\left\{-p + e^2\sin\theta\,(\cos\theta + p\sin\theta)\right\}. \qquad (1)$$

Differentiating again, we have

$$(y-k)(1-e^2\sin^2\theta) - (x-h)\,e^2\sin\theta\cos\theta$$
$$= e^2(\cos\theta + p\sin\theta)^2 - (1+p^2) \qquad (2)$$

and differentiating a third time

$$\frac{r}{q}\left\{(y-k)(1-e^2\sin^2\theta) - (x-h)\,e^2\sin\theta\cos\theta\right\}$$
$$= -3p + 3e^2\sin\theta\,(p\sin\theta + \cos\theta). \qquad (3)$$

Now $\quad \dfrac{x-h}{y-k} = \dfrac{e^2\sin\theta\,(\cos\theta + p\sin\theta) - p}{1 - e^2\cos\theta\,(\cos\theta + p\sin\theta)} \quad$ from (1).

Hence from (2)

$$(1-e^2\sin^2\theta) - e^2\sin\theta\cos\theta\left\{\frac{e^2\sin\theta\,(\cos\theta + p\sin\theta) - p}{1 - e^2\cos\theta\,(\cos\theta + p\sin\theta)}\right\}$$
$$= \left\{\frac{e^2(\cos\theta + p\sin\theta)^2 - (1+p^2)}{q}\right\}\frac{1}{y-k}.$$

Now from (3)

$$\frac{r}{q}\left\{1 - e^2\sin^2\theta - e^2\sin\theta\cos\theta\,.\,\frac{e^2\sin\theta\,(\cos\theta + p\sin\theta) - p}{1 - e^2\cos\theta\,(\cos\theta + p\sin\theta)}\right\}$$
$$\frac{-3p + 3e^2\sin\theta\,(p\sin\theta + \cos\theta)}{y-k};$$

$$\therefore \frac{r}{-}\ \frac{\left\{1 - e^2\sin^2\theta - e^2\sin\theta\cos\theta\,.\,\dfrac{e^2\sin\theta\,(\cos\theta + p\sin\theta) - p}{1 - e^2\cos\theta\,(\cos\theta + p\sin\theta)}\right\}}{-\,3p + 3e^2\sin\theta\,(p\sin\theta + \cos\theta)}$$

$$r\ \frac{\left\{1 - e^2\sin^2\theta - e^2\sin\theta\cos\theta\,.\,\dfrac{e^2\sin\theta\,(\cos\theta + p\sin\theta) - p}{1 - e^2\cos\theta(\cos\theta + p\sin\theta)}\right\}}{e^2\,(\cos\theta + p\sin\theta)^2 - (1 + p^2)}.$$

$$\therefore\ \frac{r}{q^2}\left\{e^2(\cos\theta + p\sin\theta)^2 - (1 + p^2)\right\}$$

$$= -\,3p + 3e^2\sin\theta\,(p\sin\theta + \cos\theta);$$

$$\therefore\ e^2\left\{\frac{r}{q^2}(\cos\theta + p\sin\theta)^2 - 3\sin\theta\,(p\sin\theta + \cos\theta)\right\}$$

$$= -\,3p + \frac{1 + p^2}{q^2}\,.\,r.$$

Now for θ put $90^0 + \theta$, and put $-\,e_1^2$ for e^2, then

$$-\,e_1^2\left\{\frac{r}{q^2}(p\cos\theta - \sin\theta)^2 - 3\cos\theta\,(p\cos\theta - \sin\theta)\right\}$$

$$= -\,3p + \frac{1 + p^2}{q^2}\,.\,r.$$

Add these two equations, then

$$\left\{-\,3p + \frac{1 + p^2}{q^2}\,r\right\}\left\{\frac{1}{e^2} - \frac{1}{e_1^2}\right\}$$

$$= \frac{r}{q^2}\left\{1 + p^2\right\} - 3p;$$

$$\therefore\ \frac{1}{e^2} - \frac{1}{e_1^2} = 1.$$

Now e_1 is the eccentricity of an hyperbola, which has a contact of the third order with a curve at the same point, and whose major axis is perpendicular to that of the ellipse.

Problem XV.

A conical surface will intersect an ellipsoid in two plane curves or none; and when the vertex of the surface is fixed, the intersections of the two plane sections will all lie in one plane.

Let the vertex of the cone be the origin, and the line joining the vertex and the centre of the ellipsoid the axis of z; then the plane of xy may be so assumed that the equation to the ellipsoid may take the form

$$\frac{x^2}{\alpha^2} + \frac{y^2}{\beta^2} + \left(\frac{z-d}{\gamma}\right)^2$$

α, β, γ being three semi-conjugate diameters, and d the distance of the centre from the origin.

The equation to the cone is homogeneous in this case, (Hymers' *Geometry*, Art. 163), and may take the form

$$ax^2 + by^2 + z^2 + a'yz + b'xz = 0.$$

Suppose these surfaces to be cut by a plane

$$z = Ax + By + C;$$

then the equations to the projections on the plane xy of the sections will be

$$\frac{x^2}{\alpha^2} + \frac{y^2}{\beta^2} + \left(\frac{Ax + By + C - d}{\gamma}\right)^2 = 1,$$

and $$ax^2 + by^2 + (Ax + By + C)^2 + a'y(Ax + By + C) + b'x(Ax + By + C) = 0.$$

These must be identical. Hence equating the ratios of the coefficients of x and y we have $Aa' = Bb'$: hence equating the coefficients of xy, we have $B = 0$ and then $a' = 0$.

Equating the other coefficients we have

$$\frac{1}{a^2} + \frac{A^2}{\gamma^2} = p\,(a + A^2 + b'A) \quad (1)$$

$$\frac{1}{\beta^2} = pb \quad (2)$$

$$2\,(C - d)\,\frac{A}{\gamma^2} = p\,(2AC + Cb') \quad (3)$$

$$\left(\frac{C - d}{\gamma}\right)^2 - 1 = pC^2 \quad (4)$$

Hence $$p = \frac{1}{b\beta^2}.$$

And the equations

$$\left(\frac{C - d}{\gamma}\right)^2 - 1 = \frac{C}{b\beta^2},$$

$$\frac{1}{a^2} + \frac{A^2}{\gamma^2} = \frac{1}{b\beta^2}(a + A^2 + Ab'),$$

give two possible or impossible values of A and C

Now, from (3),

$$2\,.\,(C - d)\,\frac{A}{\gamma^2} = \frac{C}{b\beta^2}(2A + b'),$$

and from (4),

$$\frac{2\,.\,(C - d)}{C}\,\frac{A}{\gamma^2} = \frac{2A + b'}{b\beta^2}$$

$$= \frac{2A}{\gamma^2}\cdot\frac{\frac{d}{b\beta^2} \pm \sqrt{\frac{d^2}{\gamma^4} + \left(1 - \frac{d^2}{\gamma^2}\right)\cdot\left(\frac{1}{\gamma^2} - \frac{1}{b\beta^2}\right)}}{\frac{d}{\gamma^2} \pm \sqrt{\frac{d^2}{\gamma^4} + \left(1 - \frac{d^2}{\gamma^2}\right)\cdot\left(\frac{1}{\gamma^2} - \frac{1}{b\beta^2}\right)}}$$

Therefore, from (1),

$$\frac{1}{a^2}+\frac{A^2}{\gamma^2}=\frac{a}{b\beta^2}-\frac{A^2}{b\beta^2}+\frac{2A^2}{\gamma^2}\cdot\left(\frac{\frac{d}{b\beta^2}\pm\sqrt{\ldots\ldots}}{\frac{d}{\gamma^2}\pm\sqrt{\ldots\ldots}}\right)$$

The two values A_1, A_2 of A are those given by the positive and negative signs in the above equation.

Let their coefficients be L_1, L_2, and let the two values of C be C_1, C_2; then the equations to the two planes are

$$\left.\begin{aligned} z &= A_1 x + C_1 \\ z &= A_2 x + C_2 \end{aligned}\right\}$$

$$\therefore\ (L_1-L_2)z=(L_1A_1-L_2A_2)x+L_1C_1-L_2C_2.$$

Now $L_1A_1=L_2A_2$, since each $=\sqrt{\dfrac{a}{b\beta^2}-\dfrac{1}{a^2}}$;

$$\therefore\ (L_1-L_2)z=L_1C_1-L_2C_2.$$

Now the only constant in the equation to the cone, involved in L_1, L_2 is b, and since $a'=0$, we see evidently that the plane xz is a diametral plane and contains the axis of the cone, and the coefficient b of y^2 will not alter as this axis alters its inclination to the axes of x and z, since the cone will remain in the same position relatively to the axis of y.

Hence the above is the equation to a fixed plane parallel to plane xy, and contains all the lines of intersection of the plane sections.

Similarly, if $A=0$ or the axis of the cone be in the plane yz, there will be a corresponding plane.

The other condition of the identity of the equations gives no result, and is therefore not to be considered.

Problem XVI.

The sun's spots, when observed through a powerful telescope, exhibit phenomena which occur in the following order. Immediately surrounding a dark spot is a light ring; next to this appears a darker, usually called the penumbra, which is bounded by a yet darker line; then again a light ring, brighter than the general body of the sun, into which the alternations of light and shade at length subside.

How does this appearance favour the hypothesis of actual eminences of the sun's body exposed to view by fluctuations in a luminous atmosphere, and seem at variance with the supposition of a dark body surrounded by a cloudy as well as a luminous atmospherical stratum?

The first of the hypotheses stated above is due to Lalande, the other to Sir William Herschell. Many ingenious conjectures have been made respecting the nature of the sun's spots. Some have supposed them to be solid bodies revolving very near its surface, others, the smoke emitted on the eruption of volcanos, or a scum floating on an ocean of fluid matter. But the fixedness of their position on the sun's surface and their regular revolution render these suppositions very improbable. Dr Wilson of Glasgow conjectured that they might be cavities in the body of the sun, seen through breaks in a luminous shell of matter not fluid (*Phil. Trans.* 1774). This opinion was however opposed by Lalande as being insufficient to explain the appearances, and he replaced it by the one given above.

It is now generally agreed that the sun's spots are actual eminences of the sun's dark body exposed to view by the agitation of a luminous atmosphere. The difference of opinion is in the method in which this disclosure takes place. Sir William Herschell (*Phil. Trans.* 1795) maintained that the luminous strata are

superposed on a cloudy stratum, beneath which is the dark body of the sun. When these strata are partially removed by fluctuations in the atmosphere, the upper more than the lower, the body of the sun which reflects no light forms the central spot, while the partial reflection from the cloudy stratum causes the penumbra. *This hypothesis however requires that the penumbra be uniform in its shade;* and all observers from Cassini and De la Hire bear testimony against this.

The other hypothesis supposes no cloudy stratum, the central spot is the dark body of the sun, the penumbra is the *shadow* of the eminence, the external bright ring is the brink of the luminous wave. The alternations of light and shade are exactly those which would accompany a shadow. This is a much better method of explaining the formation of the penumbra, than that of a cloudy stratum, or that it is the shoaling declivity of the eminence seen faintly through a thin stratum of luminous matter. This subject will be seen fully discussed in Montucla's *Histoire des Mathématiques*, Liv. IV. p. 7; *La Connaissance des Temps*, 1798; and *Mémoires de l'Académie*, 1776.

PROBLEM XVII.

Prove the following formula:

$$\Delta^r_{x=0}\, x^n = r\,\Delta^{r-1}_{x=1}\, x^{n-1} \quad {}^{-\infty}\!\!\int_t^{+\infty} t^n e^{-t^2} = \frac{\sqrt{\pi}}{2^n}\, d^n_{t=0}\, e^{t^2}.$$

Find ${}^0\!\!\int_x^{\infty} \frac{x^{m-1}}{(1+x^n)^r}$, *and thence shew that*

$$\left\{{}^0\!\!\int_x^{\infty} x^{m-1} e^{-x^n}\right\} . \left\{{}^0\!\!\int_x^{\infty} x^{n-m-1} e^{-x^n}\right\} =$$

$$n^2 \sin \frac{m\pi}{n}$$

If $G(X) = \phi(t)$ *represents the generating function of* (X), *then*

$$\frac{\phi t}{a - \log_e t} = G\left(\epsilon^{-ax} \int_x \epsilon^{ax} X\right),$$

$$\textit{and} \quad \frac{t\phi t}{1 - at} = G\left(a^{x-1} \Sigma \frac{X}{a^x}\right),$$

Apply the last two formulæ to solve by the calculus of generating functions the linear differential equation or equation of differences of the n^{th} *order when all the coefficients are constant, and the second member* $= X$.

I. By Finite Differences,

$$\Delta^r x^n = (x + r)^n - r(x + r - 1)^n + \frac{r \cdot (r - 1)}{1 \cdot 2} (x + r - 2)^n$$

$$+ \ldots\ldots \mp x^n.$$

Now let $x = 0$.

Then

$$\Delta^r_{x=0} x^n = r^n - r(r - 1)^n + \frac{r \cdot (r - 1)}{1 \cdot 2} (r - 2)^n - \&c. \pm r.$$

Again,

$$\Delta^{r-1}_x x^{n-1} = (x + r - 1)^{n-1} - (r - 1) \cdot (x + r - 2)^{n-1}$$

$$+ \frac{(r - 1) \cdot (r - 2)}{1 \cdot 2} (x + r - 3)^{n-1} - \&c. \pm x^{n-1}.$$

Now let $x = 1$.

$$\Delta^{r-1}_{x=1} x^{n-1} = r^{n-1} - (r - 1) \cdot (r - 1)^{n-1}$$

$$+ \frac{(r - 1) \cdot (r - 2)}{1 \cdot 2} (r - 2)^{n-1} - \&c. \pm 1.$$

And

$$r \Delta^{r-1}_{x=1} x^{n-1} = r^n - r(r - 1)^n + \frac{r \cdot (r - 1)}{1 \cdot 2} (r - 2)^n - \&c. \pm r;$$

$$\therefore \Delta^r_{x=0} x^n = r \Delta^{r-1}_{x=1} x^{n-1}.$$

II. Again,

$$\int_t t^n e^{-t^2} = -\tfrac{1}{2}\left\{t^{n-1}e^{-t^2} - (n-1)\int_t t^{n-2}e^{-t^2}\right\};$$

$$\therefore\ {}_{-\infty}\int_t^{+\infty} t^n e^{-t^2} = \frac{n-1}{2}\, {}_{-\infty}\int_t^{+\infty} t^{n-2}e^{-t^2}.$$

So
$$ {}_{-\infty}\int_t^{+\infty} t^{n-2}e^{-t^2} = \frac{n-3}{2}\, {}_{-\infty}\int_t^{+\infty} t^{n-2}e^{-t^2},$$

&c. = &c.

$${}_{-\infty}\int_t^{+\infty} t^2 e^{-t^2} = \tfrac{1}{2}\, {}_{-\infty}\int_t^{+\infty} \epsilon^{-t^2} = \frac{\sqrt{\pi}}{2}, \text{ when } n \text{ is even,}$$

$${}_{-\infty}\int_t^{+\infty} t\epsilon^{-t^2} = -\tfrac{1}{2}\epsilon^{-t^2}(t = \mp\infty) = 0, \quad (n \text{ odd});$$

$$\therefore\ \int_t t^n e^{-t^2} = \frac{\sqrt{\pi}}{{}_n} \,.\, (n-1)\,.\,(n-3)\ldots\ldots 3\,.\,1, \quad (n \text{ even}).$$

Now
$$\epsilon^{(t+h)^2} = \epsilon^{t^2}\,.\,\epsilon^{2th}\,.\,\epsilon^{h^2}$$

$$\epsilon^{t^2}\left\{1 + 2th + \frac{(2th)^2}{(2)} + \ldots\ldots\right\}$$

$$\times\left\{1 + h^2 + \frac{(h^2)^2}{(2)} + \ldots\ldots + \frac{(h^2)^{\frac{n}{2}}}{\left(\frac{n}{2}\right)} + \right.$$

Now $\dfrac{\epsilon^{t^2}h^n}{(n\backslash}$ is the only term involving h^n in the second member which does not vanish when $t = 0$;

$$\therefore\ d^n_{t=0}\,\epsilon^{t^2} = \frac{1\,.\,2\,.\,3\ldots\ldots n}{1\,.\,2\,.\,3\ldots\ldots\frac{n}{2}}$$

$$= 2^{\frac{n}{2}}\,.\,1\,.\,3\,.\,5\ldots\ldots(n-1), \quad (n \text{ even});$$

and $d^n_{t=0}\epsilon^{t^2} = 0$, when n is odd;

$$\therefore\ {}_{-\infty}\int_t^{+\infty} t^n \epsilon^{-t^2} = \frac{\sqrt{\pi}}{2^n}\, d^n_{t=0}\,\epsilon^{t^2}.$$

III. Again, to reduce the integral ${}^0\!\int_x^\infty \frac{x^{m-1}}{(1+x^n)^r}$.

assume $$P = \frac{}{(1+x^n)^{r-1}}$$

Then $$d_x P = \frac{x^{m-1}}{(1+x^n)^r}\{m(1+x^n) - (r-1)nx^n\}$$

$$= (m - nr + n)\cdot\frac{x^{m-1}}{(1+x^n)^{r-1}} + n.(r-1).\frac{x^{m-1}}{(1+x^n)^r};$$

$$\therefore\ {}^0\!\int_x^\infty \frac{x^{m-1}}{(1+x^n)^r} = \frac{nr-m-n}{n(r-1)}\,{}^0\!\int_x^\infty \frac{x^{m-1}}{(1+x^n)^{r-1}}.$$

So $${}^0\!\int_x^\infty \frac{x^{m-1}}{(1+x^n)^{r-1}} = \frac{nr-m-2n}{n(r-2)}\,{}^0\!\int_x^\infty \frac{x^{m-1}}{(1+x^n)^{r-2}},$$

&c. = &c.

$${}^0\!\int_x^\infty \frac{x^{m-1}}{(1+x^n)^2} = \frac{n-m}{n}\,{}^0\!\int_x^\infty \frac{x^{m-1}}{1+x^n}$$

$$\frac{n-m}{n}\quad\frac{\pi}{n\sin\frac{m\pi}{n}}$$

therefore, collecting these results,

$${}^0\!\int_x^\infty \frac{x^{m-1}}{(1+x^n)^r}\quad \frac{n(r-1)-m}{n(r-1)} \times \frac{n(r-2)-m}{n(r-2)} \times \ldots$$

$$\times\frac{n-m}{n}\cdot\frac{\pi}{n\sin\frac{m\pi}{n}}.$$

IV. Again, since

$${}^0\!\int_y^\infty \frac{y^{m-1}}{(1+y^n)^r} = \left\{\frac{n-m}{n}\times\frac{2n-m}{2n}\times\frac{3n-m}{3n}\times\ldots\right.$$

$$\left.\times\frac{n(r-1)-m}{n(r-1)}\right\}\times\frac{\pi}{n\sin\frac{m\pi}{n}}$$

Changing m into $n-m$, we have

$$ {}^{0}\!\int_{y}^{\infty}\frac{y^{n-m-1}}{(1+y^{n})^{r}} = \left\{\frac{m}{n}\times\frac{n+m}{2n}\times\frac{2n+m}{3n}\times\ldots\right. $$

$$ \left.\times\frac{n(r-2)+m}{n(r-1)}\right\}\frac{\pi}{n\sin\frac{m\pi}{n}} $$

Now let $\qquad y^{n}=\frac{x^{n}}{r} \qquad$ or $y=\frac{x}{r^{\frac{1}{n}}}$.

Then $$ {}^{0}\!\int_{y}^{\infty}\frac{y^{m-1}}{(1+y^{n})^{r}} = \frac{1}{r^{\frac{m}{n}}}\,{}^{0}\!\int_{x}^{\infty}\frac{x^{m-1}}{\left\{1+\frac{x^{n}}{r}\right\}^{r}}, $$

and $$ {}^{0}\!\int_{y}^{\infty}\frac{y^{n-m-1}}{(1+y^{n})^{r}} = \frac{1}{r^{\frac{n-m}{n}}}\,{}^{0}\!\int_{x}^{\infty}\frac{x^{n-m-1}}{\left\{1+\frac{x^{n}}{r}\right\}^{r}}. $$

Now multiply these together, and then make r infinite; then, since $\left(1+\frac{x^{n}}{r}\right)^{r}=\epsilon^{x^{n}}$ in this case, we shall have

$$ \left({}^{0}\!\int_{x}^{\infty}x^{m-1}\epsilon^{-x^{n}}\right).\left({}^{0}\!\int_{x}^{\infty}x^{n-m-1}\epsilon^{-x^{n}}\right) = {}^{0}\!\int_{y}^{\infty}\frac{y^{m-1}}{(1+y^{n})^{r}}\times{}^{0}\!\int_{y}^{\infty}\frac{y^{n}}{(1+y^{n})^{r}} $$

$$ \frac{n-m}{n}\,\frac{n+m}{n}\,\frac{3n-m}{3n}\,\frac{3n+m}{3n}\,.\,\&\text{c}. $$

$$ \times\frac{m}{n}.\frac{2n-m}{2n}.\frac{2n+m}{2n}.\frac{4n-m}{4n}\,\frac{4n+m}{4n}.\,\&\text{c}. \qquad \frac{m\pi}{n} $$

Now $$ \cos\frac{m\pi}{2n} = \frac{n-m}{n}.\frac{n+m}{n}.\frac{3n-m}{3n}.\frac{3n+m}{3n}.\,\&\text{c}. $$

$$ \frac{2}{\pi}.\sin\frac{m\pi}{2n} = \frac{m}{n}.\frac{2n-m}{2n}.\frac{2n+m}{2n}.\frac{4n-m}{4n}.\,\&\text{c}. $$

by the usual expansions;

$$\therefore\ \left({}^0\!\!\int_x^\infty x^{m-1}\epsilon^{-x^n}\right).\left({}^0\!\!\int_x^\infty x^{n-m-1}\epsilon^{-x^n}\right)$$

$$= \frac{2}{\pi}.\sin\frac{m\pi}{2n}.\cos\frac{m\pi}{2n}.\frac{\pi^2}{n^2\sin^2\frac{m\pi}{n}}$$

$$= \frac{\pi}{n^2\sin\frac{m\pi}{n}}.$$

There is, however, another method of arriving at this last expression.

Let $\quad Q = \left({}^0\!\!\int_x^\infty x^{m-1}\epsilon^{-x^n}\right).\left({}^0\!\!\int_y^\infty y^{n-m-1}\epsilon^{-y^n}\right)$

$$= {}^0\!\!\int_x^\infty {}^0\!\!\int_y^\infty \left\{x^{m-1}.y^{n-m-1}.\epsilon^{-(x^n+y^n)}\right\}.$$

Now let $\quad y = ux.$

Then $\quad Q = {}^0\!\!\int_x^\infty {}^0\!\!\int_u^\infty x^{n-1}.\epsilon^{-(1+u^n)x^n}.u^{n-m-1}$

$$-\frac{1}{n}\,{}^0\!\!\int_u^\infty \frac{u^{n-m-1}}{1+u^n}.\epsilon^{-(1+u^n)x^n},\quad \begin{pmatrix} x=0 \\ x=\infty \end{pmatrix},$$

$$\frac{1}{n}\,{}^0\!\!\int_u^\infty \frac{u^{n-m-1}}{1+u^n}$$

$$\frac{\pi}{n^2\sin\frac{m\pi}{n}}$$

V. $\quad \int_x \epsilon^{ax}X = \frac{1}{a}\epsilon^{ax}X - \frac{1}{a}\int_x\epsilon^{ax}d_xX.$

So $\quad \int_x\epsilon^{ax}d_xX = \frac{1}{a}\epsilon^{ax}d_xX - \frac{1}{a}\int_x\epsilon^{ax}d_x^2X,$

&c. = &c.

$$\therefore\ \int_x\epsilon^{ax}X = \frac{\epsilon^{ax}}{a}\left\{X - \frac{1}{a}d_xX + \frac{1}{a^2}d_x^2X - \text{\&c.}\right\};$$

$$\therefore\ \epsilon^{-ax}\int_x\epsilon^{ax}X = \frac{1}{a}\left(X - \frac{d_xX}{a} + \frac{d_x^2X}{a^2} - \text{\&c.}\right);$$

$$G\left(\epsilon^{-ax}\int_x \epsilon^{ax} X\right) = \frac{1}{a}\left(GX - \frac{1}{a}\,Gd_x X + \frac{1}{a^2}\,Gd_x^2 X - \text{\&c.}\right)$$

$$= \frac{\phi t}{a}\left\{1 - \frac{1}{a}\log_e \frac{1}{t} + \frac{1}{a^2}\left(\log_e \frac{1}{t}\right)^2 - \text{\&c.}\right\}$$

$$= \frac{\phi t}{a}\left\{1 + \frac{1}{a}\log_e t + \frac{1}{a^2}(\log_e t)^2 + \text{\&c.}\right\}$$

$$= \frac{\phi t}{a} \cdot \frac{1}{1 - \frac{1}{a}\log_e t}.$$

$$\therefore\ G\left(\epsilon^{-ax}\int_x \epsilon^{ax} X\right) = \frac{\phi(t)}{a - \log_e t}.$$

VI. Again,

$$\Sigma(u_x v_x) = u_x \Sigma v_x - \Delta u_x \Sigma^2 v_{x+1} + \Delta^2 u_x \Sigma^3 v_{x+2} - \text{\&c.}$$

Let $\qquad v_x = a^{-x} \qquad u_x = X.$

$$\Sigma(u_x v_x) = X \cdot \frac{a^{-x}}{a^{-1} - 1} - \Delta X \cdot \frac{a^{-x-1}}{(a^{-1} - 1)^2} + \text{\&c.};$$

$$a^{x-1}\Sigma\left(\frac{X}{a^x}\right) = X \cdot \frac{a^{-1}}{a^{-1} - 1} - \Delta X \cdot \frac{a^{-2}}{(a^{-1} - 1)^2} + \text{\&c.}$$

$$= \frac{X}{1 - a} - \Delta X \cdot \frac{1}{(1 - a)^2} + \Delta^2 X \cdot \frac{1}{(1 - a)^3} - \text{\&c.};$$

$$G\left(a^{x-1}\Sigma\frac{X}{a^x}\right) = -\phi(t)\left\{\frac{1}{a - 1} + \frac{1 - t}{t} \cdot \frac{1}{(a - 1)^2} + \left(\frac{1 - t}{t}\right)^2 \cdot \frac{1}{(a - 1)^3} + \dots\right\}$$

$$= \frac{-\phi(t)}{a - 1} \cdot \left\{1 + \frac{1 - t}{t} \cdot \frac{1}{a - 1} + \left(\frac{1 - t}{t}\right)^2 \cdot \frac{1}{(a - 1)^2} + \dots\right\}$$

$$= \frac{-\phi(t)}{a - 1} \cdot \frac{1}{1 - \frac{1 - t}{at - t}}.$$

$$= \frac{-\phi(t)}{a-1} \cdot \frac{at-t}{at-1}$$

$$= \frac{t\phi t}{1-at}.$$

VII. Suppose that the linear differential equation is

$$d_x^n y + p_1 d_x^{n-1} y + p_2 d_x^{n-2} y + \dots + p_n y = X,$$

and suppose that the roots of the auxiliary equation

$$a^n + p_1 a^{n-1} + p_2 a^{n-2} + \dots + p_n = 0$$

are $a_1, a_2 \dots a_n$, then its solution is

$$y = \epsilon^{-a_1 x} \int_x \epsilon^{(a_1 - a_2)x} \int_x \epsilon^{(a_2 - a_3)x} \dots \int_x \epsilon^{a_n x} X.$$

Suppose that

$$\frac{\phi t}{a_n - \log_\epsilon t} = G\{\epsilon^{-a_n x} \int_x \epsilon^{a_n x} X\} = \psi(t) = GX'.$$

Then $G\{\epsilon^{-a_{n-1} x} \int_x \epsilon^{a_{n-1}} X'\} = \dfrac{\psi t}{a_{n-1} - \log_\epsilon t},$

or $G\{\epsilon^{-a_{n-1}} \int_x \epsilon^{(a_{n-1} - a_n)x} \int_x \epsilon^{ax} X\}$

$$= \phi(t) \cdot \frac{1}{a_n - \log_\epsilon t} \cdot \frac{1}{a_{n-1} - \log_\epsilon t}$$

$$= \chi(t) = GX''.$$

Then $G(\epsilon^{-a_{n-2}x} \int_x \epsilon^{a_{n-2}x} X'') = \dfrac{\chi t}{a_{n-2} - \log_\epsilon t},$

or $G\{\epsilon^{-a_{n-2}} \int_x \epsilon^{(a_{n-2} - a_{n-1})x} \int_x \epsilon^{(a_{n-1} - a_n)x} \int_x \epsilon^{a_n x} X\}$

$$= \phi(t) \cdot \frac{1}{a_n - \log_\epsilon t} \cdot \frac{1}{a_{n-1} - \log_\epsilon t} \cdot \frac{1}{a_{n-2} - \log_\epsilon t},$$

and so on;

$$\therefore \; G(y) \quad \frac{\phi(t)}{(a_n - \log_\epsilon t) \cdot (a_{n-1} - \log_\epsilon t) \dots (a_1 - \log_\epsilon t)}.$$

Hence when $\phi(t)$ is known, we can expand this in powers of t, and the coefficient of t^x will be the value of y.

VIII. Lastly, suppose the linear equation of differences to be

$$u_{x+n} + p_1 u_{x+n-1} + p_2 u_{x+n-2} + \ldots + p_n u_x = X.$$

Then if a_1, $a_2 \ldots a_n$ be the roots of the auxiliary equation,

$$a^n + p_1 a^{n-1} + \ldots + p_n = 0.$$

The value of u_x is

$$u_x = \frac{(-1)^n}{p_n} a_1{}^x \Sigma \left(\frac{a_2}{a_1}\right)^x . \Sigma \left(\frac{a_3}{a_2}\right)^x \ldots \Sigma \frac{X}{a_n{}^x}.$$

Now let $G\left(a_n{}^x \Sigma \dfrac{X}{a_n{}^x}\right) = \dfrac{a_n t \phi(t)}{1 - a_n t} = G(X') = \psi(t).$

Then $G\left(a_{n-1}^x \Sigma \dfrac{X'}{a_{n-1}^x}\right) = \dfrac{a_{n-1} t \psi t}{1 - a_{n-1} t},$

or $G\left\{a_{n-1}^x \Sigma \left(\dfrac{a_n}{a_{n-1}}\right)^x \Sigma \dfrac{X}{a_n^x}\right\} \quad \dfrac{a_n t . a_{n-1} t . \phi t}{(1 - a_n t)(1 - a_{n-1} t)}.$

So $G\left\{a_{n-2}^x \Sigma \left(\dfrac{a_{n-1}}{a_{n-2}}\right)^x \Sigma \left(\dfrac{a_n}{a_{n-1}}\right)^x \Sigma \dfrac{X}{a_n^x}\right\}$

$$\frac{a_n t . a_{n-1} t . a_{n-2} t . \phi t}{(1 - a_n t) . (1 - a_{n-1} t) . (1 - a_{n-2} t)},$$

and at last

$$G\left\{a_1^x \Sigma \left(\frac{a_2}{a_1}\right)^x \ldots \Sigma \frac{X}{a_n^x}\right\} = \frac{a_n . a_{n-1} \ldots a_1 t^n \phi t}{(1 - a_n t) \ldots (1 - a_1 t)}$$

$$(-1)^n . p_n . \frac{t^n \phi t}{(1 - a_n t) \ldots (1 - a_1 t)},$$

and $$Gu_s = \frac{t^n \phi t}{(1 - a_n t) \cdot (1 - a_{n-1} t) \ldots (1 - a_1 t)},$$

and taking the coefficient of t^x in the expansion of this, we obtain the value of u_x.

Problem XVIII.

Determine the motion of two weights connected by an elastic string, and moving along the convex surface of a cycloid having its axis vertical.

Fig. 22. Let P, Q be the weights moving on the cycloid whose axis AB is vertical, $AP = s$, $AQ = s_1$, $AM = x$, $AN = x_1$, a the radius of the generating circle, T the tension of the string.

The equations of motion are

$$d_t^2 s = \frac{T}{P} + g \cdot \frac{x}{AS} = \frac{T}{P} + g \cdot \frac{2AS}{4a}$$

$$\left. \begin{aligned} &= \frac{T}{P} + g \cdot \frac{s}{4a} \\ \text{and } d_t^2 s_1 &= -\frac{T}{Q} + g \cdot \frac{s_1}{4a} \end{aligned} \right\}$$

$$\therefore\ d_t^2 (s_1 - s) = -T\left(\frac{1}{P} + \frac{1}{Q}\right) + \frac{g}{4a}(s_1 - s).$$

Now the string being elastic, if l be its original length, and ϵ the quantity by which a unit of length of the string, when subjected to the unit of tension, would be stretched, then

$$s_1 - s = l + l\epsilon T,$$

and $$T = \frac{s_1 - }{l\epsilon}$$

$$\therefore\ d_t^2(s_1 - s) = -\left(\frac{1}{P} + \frac{1}{Q}\right)\cdot\left(\frac{s_1 - s}{l\epsilon}\right) + \left(\frac{1}{P} + \frac{1}{Q}\right)\frac{1}{\epsilon}$$
$$+ \frac{g}{4a}(s_1 - s),$$

or $d_t^2(s_1 - s) + \left\{\left(\frac{1}{P} + \frac{1}{Q}\right)\frac{1}{l\epsilon} - \frac{g}{4a}\right\}(s_1 - s) = \left(\frac{1}{P} + \frac{1}{Q}\right)\frac{1}{\epsilon}$,

which is a linear equation of the second order, and may be always solved.

If
$$m = \left(\frac{1}{P} + \frac{1}{Q}\right)\frac{1}{l\epsilon} - \frac{g}{4a},$$
$$n = \left(\frac{1}{P} + \frac{1}{Q}\right)\frac{1}{\epsilon},$$

the solution is

$$s_1 - s = \frac{n}{m} + c_1 e^{t\sqrt{-m}} + c_2 e^{-t\sqrt{-m}},$$

which if the coefficient of the second term of the differential equation be positive, may be reduced to the form

$$s_1 - s = \frac{n}{m} + A\cos(t\sqrt{m} + B)^*.$$

However, taking the general case,

$$s_1 = s + \frac{n}{m} + c_1 e^{t\sqrt{-m}} + c_2 e^{-t\sqrt{-m}},$$

whence c_1, c_2 are determined by the conditions that when $t = 0$, $s_1 - s$ and $d_t s_1 - d_t s$ are given quantities.

Now, from the original equations,

$$Pd_t^2 s + Qd_t^2 s_1 = \frac{g}{4a}(Ps + Qs_1),$$

or substituting for $d_t^2 s_1$ in terms of $d_t^2 s$,

$$Pd_t^2 s + Q\{d_t^2 s + n - m(s_1 - s)\} = \frac{g}{4a}(Ps + Qs_1),$$

* This would indicate that the length of PQ made oscillations between its extreme values in the time $\frac{\pi}{\sqrt{m}}$.

or after substitution for s_1,

$$d_t^2 s - \frac{g}{4a} s - \frac{1}{Pl\epsilon}\left(c_1 e^{t\sqrt{-m}} + c_2 e^{-t\sqrt{-m}}\right) = \frac{Q}{P+Q} \cdot \frac{g}{4a} \cdot \frac{n}{m},$$

$$\text{or if } y = s + \frac{Q}{P+Q} \cdot \frac{n}{m},$$

$$d_t^2 y - \frac{g}{4a} y = T, \quad \text{a function of } t.$$

If $\frac{1}{2}\sqrt{\frac{g}{a}} = h$, the solution is

$$y = e^{-ht} \int_t e^{2ht} \int_t e^{-ht}\, T,$$

or after performing the operations,

$$s + \frac{n}{m} \cdot \frac{Q}{P+Q} = a_1 e^{\frac{1}{2}\sqrt{\frac{g}{a}} \cdot t} + a_2 e^{-\frac{1}{2}\sqrt{\frac{g}{a}} \cdot t}$$

$$- \frac{Q}{P+Q}\left\{c_1 e^{t\sqrt{-m}} + c_2 e^{-t\sqrt{-m}}\right\},$$

which gives the space described in a given time by P: similarly with regard to Q. The constants a_1, a_2 are known from given values of $d_t s$ and s when $t = 0$.

Problem XIX.

Find the nature of the curve in which every chord subtending a given angle α *at a fixed point* A *shall also subtend a given angle* β *at a fixed point* B. *Shew also that* $\rho \cos \mathrm{m}\theta = \mathrm{a} \cos (\mathrm{m} \pm 1)\theta$ *is a particular solution where* $\mathrm{AB} = \mathrm{a}$, *and* m *is a constant depending on the values of* α *and* β.

Let PQ be the chord;

$$QAB = \theta, \quad PBA = \phi, \quad PAQ = \alpha, \quad PBQ = \beta, \quad AQ = \rho.$$

Then if $AQ = u_\theta$, $AP = u_{\theta+\alpha}$,

$$\text{and } \frac{u_\theta}{a} = \frac{\sin(\phi +}{\sin(\phi + \beta + \theta)},$$

$$\frac{u_{\theta+\alpha}}{a} = \frac{\sin\phi}{\sin(\phi + \theta + \alpha)};$$

therefore eliminating $\tan\phi$, we have

$$\frac{u_{\theta+\alpha}\cos(\theta+\alpha) - a}{u_\theta\cos(\theta+\beta) - a\cos\beta} = \frac{u_{\theta+\alpha}\sin(\theta+\alpha)}{u_\theta\sin(\theta+\beta) - a\sin\beta},$$

or

$$u_\theta \cdot u_{\theta+\alpha}\sin(\alpha-\beta) - a u_{\theta+\alpha}\sin(\theta+\alpha-\beta) + a u_\theta\sin(\theta+\beta)$$
$$= a^2\sin\beta.$$

Let $\theta = \alpha x$, $u_\theta = v_x$, then $u_{\theta+\alpha} = v_{x+1}$.

Then

$$v_x \cdot v_{x+1}\sin(\alpha-\beta) - a v_{x+1}\sin(\alpha x+\alpha-\beta) + a v_x\sin(\alpha x+\beta)$$
$$= a^2\sin$$

and

$$v_{x+1}\{v_x\sin(\alpha-\beta) - a\sin(\alpha x+\alpha-\beta)\} + a v_x\sin(\alpha x +$$
$$= a^2\sin\beta.$$

Let $v_x\sin(\alpha-\beta) - a\sin(\alpha x+\alpha-\beta) \quad \frac{w_{x+}}{w_x}$

$$\text{Then } a\left\{\sin(\alpha x+2\alpha-\beta) + \frac{w_{x+2}}{w_{x+1}}\right\}\frac{1}{\sin(\alpha-\beta)}\cdot\frac{w_{x+}}{w_x}$$

$$+ a\frac{\sin(\alpha x+\beta)}{\sin(\alpha-\beta)}\left\{a\sin(\alpha x+\alpha-\beta) + \frac{w_{x+1}}{w_x}\right\} = a^2\sin$$

$$\text{or } w_{x+2} + 2a\cos(\alpha - \beta)\sin(\alpha x + \alpha) w_{x+1}$$
$$+ a^2 \sin\alpha x \sin(\alpha x + \alpha) w_x = 0.$$

Hence, (Herschell's *Examples*),

$$w_x = \sin\alpha \sin 2\alpha \ldots \sin(x-1)\alpha \times$$
$$(-a)^x \{c_1 \cos(\alpha - \beta)x + c_2 \sin(\alpha - \beta)x\};$$

therefore,

$$\frac{w_{x+1}}{w_x} = -a\sin\alpha x \cdot \frac{c_1\cos(\alpha-\beta)(x+1) + c_2\sin(\alpha-\beta)(x+1)}{c_1\cos(\alpha-\beta)x + c_2\sin(\alpha-\beta)x};$$

$$\therefore\ v_x \sin(\alpha - \quad = a\sin(\alpha x + \alpha - \beta)$$

$$-a\sin\alpha x \cdot\cdot \frac{\cos(\alpha-\beta)(x+1) + \frac{c_2}{c_1}\sin(\alpha-\beta)(x+1)}{\cos(\alpha-\beta)x + \frac{c_2}{c_1}\sin(\alpha-\beta)x}$$

Now $\frac{c_2}{c_1}$ may be any function of x which does not change when $x+1$ is put for x, or which satisfies the equation $\phi(x) = \phi(x+1)$: substituting and reducing, we have

$$v_x\{\cos(\alpha-\beta)x + \phi(x)\sin(\alpha-\beta)x\}$$
$$= a[\cos\{(\alpha-\beta)x \pm \alpha x\} + \phi(x)\sin\{(\alpha-\beta)x \pm \alpha x\}]$$

$+$ or $-$ according as α is greater or less than β.

This, when the value of x in terms of θ is substituted and ρ for v_x, is the general solution. A particular case is when $\phi(x) = 0$, or generally $\tan(2\lambda \pm 1)\pi x$. Suppose the former; then

$$\rho\cos(\alpha-\beta)x = a\cos\{(\alpha-\beta)x \pm \alpha x\}.$$

Let m denote the arithmetical value of $\frac{\alpha-\beta}{\alpha}$.

and [illegible] $\left(\frac{\alpha - \beta}{\alpha} \pm 1\right)$

$$= \theta(m \pm 1),$$

and $(\alpha - \beta)x = \alpha x . \quad \alpha - \beta$

$$\therefore \ \rho \cos m\theta = a \cos(m \pm 1)\theta.$$

Problem XX.

A regular tetrahedron is moveable round a fixed vertical axis passing through two rings at the extremities of one of its edges; determine the least angular motion which will prevent the tetrahedron from sliding down the axis, and the coefficient of friction when the whole pressure upon the axis takes place at the upper ring.

Fig. 23. Let LA be the vertical edge, acb a section of the tetrahedron parallel to the base, $La = z$, $ad = x$, $dG = y$ the oblique co-ordinates of an element of the section, DEG a spherical triangle referred to a as centre, GM perpendicular to La.

Let $\qquad LaG = \delta, \qquad GaE = \theta,$

ϕ the inclination of two plane faces,

I .... of La to the plane abc,

w the angular velocity which is just sufficient to prevent sliding, a = an edge of the tetrahedron, R, R' the reactions of the rings.

Then from the spherical triangle DEG,

$$\cos\delta = \tfrac{1}{2}\cos\theta + \frac{\sqrt{3}}{2}\sin\theta\cos\phi,$$

and from the triangle DEH, $\cos\phi = \frac{1}{3}$,

$$\text{also } \tan\theta = \frac{y\sqrt{3}}{2x - y} \text{ from triangle } EaG;$$

$$\therefore\ \cos\delta = \frac{1}{2} \cdot \frac{1}{\sqrt{1 + \dfrac{3y^2}{(2x-y)^2}}} \times \left(1 + \frac{3y}{2x - y}\cos\phi\right)$$

$$= \frac{1}{2} \cdot \frac{x}{\sqrt{x^2 - xy + y^2}},$$

$$\text{and } \sin^2\delta = \frac{3x^2 - 4xy + 4y^2}{4(x^2 - xy + y^2)}.$$

Now mass of an element at $G = \delta x \delta y \cdot \dfrac{\sqrt{3}}{2} \cdot \delta z \sin I$,

centrifugal force of this $= \dfrac{\sqrt{3}}{2} \sin I \cdot \delta x \delta y \delta z \cdot aG \sin\delta \times w^2$

$$= \frac{w^2\sqrt{3}}{2} \sin I \cdot \delta x \delta y \delta z \sqrt{\left(y - \frac{x}{2}\right)^2 + \frac{x^2}{2}};$$

therefore whole centrifugal force

$$= \frac{w^2\sqrt{3}}{2} \sin I \int_z \int_x \left\{ \frac{y - \frac{x}{2}}{2} \sqrt{\left(y - \frac{x}{2}\right)^2 + \frac{x^2}{2}} \right.$$

$$\left. + \frac{x^2}{4} \log_e \left\{ y - \frac{x}{2} + \sqrt{\left(y - \frac{x}{2}\right)^2 + \frac{x^2}{2}} \right\} \right.$$

$$\left(\text{between limits } \begin{matrix} y = 0 \\ y = x \end{matrix}\right),$$

$$= \frac{w^2\sqrt{3}}{2} \sin I \int \int_z \left\{ \frac{x^2\sqrt{3}}{4} + \frac{x^2}{4} \log_e \left(\frac{\sqrt{3} + 1}{\sqrt{3} - 1} \right) \right\},$$

$$\text{between limits } \begin{pmatrix} x = 0 \\ x = z \end{pmatrix}, \quad \begin{pmatrix} z = 0 \\ z = a \end{pmatrix},$$

$$= \frac{w^2 a^4}{2\sqrt{3}} \cdot \frac{\sin I}{16} \left\{ \sqrt{3} + \log_e \left(\frac{\sqrt{3} + 1}{\sqrt{3} - 1} \right) \right\}.$$

Now this being altogether perpendicular to the edge, balances the reactions of the rings;

$$\therefore\ R + R' = \frac{w^2 a^4}{2\sqrt{3}} \cdot \frac{\sin I}{16}\left\{\sqrt{3} + \log_e\left(\frac{\sqrt{3}+1}{\sqrt{3}-1}\right)\right\},$$

and the friction just balances the weight. Hence

$$\mu(R + R') = \frac{g a^3}{6\sqrt{2}}$$

$$\therefore\ \frac{1}{\mu} = \frac{w^2\sqrt{3}}{2\sqrt{2}}\,\frac{a}{g}\,\frac{\sin I}{4}\left\{\sqrt{3} + \log_e\left(\frac{\sqrt{3}+1}{\sqrt{3}-1}\right)\right\},$$

and since $\sin I = \sqrt{\frac{2}{3}}$,

$$\frac{1}{w^2} = \frac{\mu}{8}\,\frac{a}{g}\left\{\sqrt{3} + \log_e\frac{\sqrt{3}+1}{\sqrt{3}-1}\right\}\ldots\ldots(a).$$

II. Again, if there be no pressure at the lower ring, the moments of the weight and centrifugal force about the upper ring are equal.

Now moment of an element

$$w^2\sqrt{3}\,\delta x\,\delta y\,\delta z \sin I\sqrt{\left(y-\frac{x}{2}\right)^2 + \frac{x^2}{2}}\,(z - aG\,.\cos\delta).$$

Hence whole moment of centrifugal force

$$= \frac{w^2\sqrt{3}}{2}\sin I\int_x\int_y\int_z\sqrt{\left(y-\frac{x}{2}\right)^2+\frac{x^2}{2}}\left(z-\frac{x}{2}\right)$$

$$= \frac{w^2\sin I\sqrt{3}}{2}\int_z\int_x\int_y\sqrt{\left(y-\frac{x}{2}\right)^2+\frac{x^2}{2}}\times z$$

$$-\frac{w^2\sqrt{3}}{4}a\sin I\int_z\int_x\int_y x\sqrt{\left(y-\frac{x}{2}\right)^2+\frac{x^2}{2}}$$

$$= \frac{w^2 \sin I \sqrt{3}}{2} \cdot \frac{a^2}{5} \cdot \left\{ \frac{1}{3} \frac{a^3 \sqrt{3}}{4} + \frac{a^3}{12} \log_e \left(\frac{\sqrt{3}+1}{\sqrt{3}-1} \right) \right\}$$

$$= \frac{w^2 \sin I \sqrt{3}}{4} \cdot \frac{a^5}{80} \left\{ \sqrt{3} + \log_e \left(\frac{\sqrt{3}+1}{\sqrt{3}-1} \right) \right\}$$

$$= \frac{w^2 \sin I \sqrt{3}\, a^5}{192} \left\{ \sqrt{3} + \log_e \left(\frac{\sqrt{3}+1}{\sqrt{3}-1} \right) \right\}.$$

Now the perpendicular from the centre of gravity on the axis $= \frac{}{2\sqrt{2}}$

$$\therefore \text{ moment of weight } = \frac{ga^4}{24}.$$

$$\therefore \frac{ga^4}{24} = \frac{w^2 \sin I \sqrt{3} a^5}{192} \left\{ \sqrt{3} + \log_e \left(\frac{\sqrt{3}+1}{\sqrt{3}-1} \right) \right\} \ldots\ldots (a).$$

Again, from the mechanical equations, putting $R' = 0$,

$$\frac{1}{w^2} = \frac{\mu}{8} \cdot \frac{a}{g} \left\{ \sqrt{3} + \log_e \left(\frac{\sqrt{3}+1}{\sqrt{3}-1} \right) \right\};$$

$$\frac{g}{aw^2} = \frac{\mu}{8} \left\{ \sqrt{3} + \log_e \left(\frac{\sqrt{3}+1}{\sqrt{3}-1} \right) \right\};$$

$$\therefore \frac{\mu}{8} = \frac{\sin I . \sqrt{3}}{8}.$$

$$\therefore \mu = \sqrt{2}.$$

This gives the value of μ, that there may be no pressure on the lower ring.

Problem XXI.

The sum of the squares of the projections of any three conjugate diameters of an ellipsoid (whose semi-axes are a, b, c) upon a given principal diameter is constant: and the tangent planes, at the extremities of three conjugate diameters, intersect in an ellipsoid whose equation is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 3.$$

We shall first prove the following property of the ellipse, that the sum of the squares of the perpendiculars let fall from the extremities of two semi-conjugate diameters on *any* diameter, is constant.

Let $x_0, y_0;\ -x_1, y_1$ (Fig. 24) be the co-ordinates of the extremities of any system CP, CD, or $a'\,b'$ of conjugate diameters, CL any other diameter,

$$PCA = \phi, \quad LCA = \theta, \quad DCa = \psi.$$

Then $Pk^2 = a'^2 \sin^2(\phi - \theta)$

$$= a'^2(\sin^2\phi\cos^2\theta + \cos^2\phi\sin^2\theta - 2\sin\theta\sin\phi\cos\theta\cos\phi),$$

$$DS^2 = b'^2(\sin^2\psi\cos^2\theta + \cos^2\psi\sin^2\theta + 2\sin\theta\sin\psi\cos\theta\cos\psi);$$

$\therefore\ Pk^2 + DS^2$

$$= b^2\cos^2\theta + a^2\sin^2\theta - 2\sin\theta\cos\theta(a'^2\sin\phi\cos\phi - b'^2\sin\psi\cos\psi),$$

because $a'^2\sin^2\phi + b'^2\sin^2\psi = b^2$;

$$\therefore\ Pk^2 + DS^2 = \frac{a^2b^2}{CL^2} - 2\sin\theta\cos\theta\,(y_0x_0 + y_1x_1).$$

Now $y_1 = \dfrac{bx_0}{a}$, and $x_1 = -\dfrac{ay_0}{b}$.

$$y_1x_1 = -y_0x_0;$$

$$\therefore\ Pk^2 + DS^2 = \frac{a^2b^2}{CL^2},$$

which is constant for all systems of conjugate diameters.

Now (Fig. 25) let Ox, Oy, Oz be three principal diameters of the ellipsoid, Ox', Oy', Oz' three semi-conjugate diameters. Suppose the plane of $x'y'$ to intersect the plane AOH in OB at an angle θ, and the plane $z'OZ$ to intersect the planes $x'y'$, and AOH in OF, OC, then the systems (Oz', OF) (OB, OC) are conjugate.

Let $x_0 y_0 z_0$, $x_1 y_1 z_1$, $x_2 y_2 z_2$ be the co-ordinates of the extremities of the three semi-conjugates $(a' b' c')$.

Then $z_0 = a' \cos x'z = a' \sin x'B \sin\theta$ ($\because$ $zB = 90^0$),

$$z_1 = b' \cos y'z = b' \sin y'B \sin\theta,$$

$$z_2 = c' \cos z'z;$$

$$\therefore\ z_0^2 + z_1^2 + z_2^2 = \sin^2\theta\,(a'^2 \sin^2 x'B + b'^2 \sin^2 y B) + c'^2 \cos^2 z'z.$$

Now $\sin\theta = \dfrac{\sin CF}{\sin FB}$,

and $a'^2 \sin^2 x'B + b'^2 \sin^2 y'B = \dfrac{OB^2 . OF^2 . \sin^2 FB}{OB^2}$

$$= OF^2 . \sin^2 FB;$$

$$\therefore\ z_0^2 + z_1^2 + z_2^2 = OF^2 \sin^2 CF + c'^2 \cos^2 z'z = c^2,$$

by a property of the ellipse previously mentioned.

We may write the above $Z^2 = c^2$.

Similarly $X^2 = a^2$, $Y^2 = b^2$.

II. Again, the equations to the tangent planes are

$$\left.\begin{aligned} \frac{xx_0}{a^2} + \frac{yy_0}{b^2} + \frac{zz_0}{c^2} &= 1 \\ \frac{xx_1}{a^2} + \frac{yy_1}{b^2} + \frac{zz_1}{c^2} &= 1 \\ \frac{xx_2}{a^2} + \frac{yy_2}{b^2} + \frac{zz_2}{c^2} &= 1 \end{aligned}\right\} \ldots\ldots(1).$$

Square these and add them, taking notice that

$$x_0^2 + x_1^2 + x_2^2 = a^2.$$

Then

$$\left.\begin{aligned}\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} &+ \frac{2xy}{a^2b^2}(x_0y_0 + x_1y_1 + x_2y_2) \\ &+ \frac{2xz}{a^2c^2}(x_0z_0 + x_1z_1 + x_2z_2) \\ &+ \frac{2yz}{b^2c^2}(y_0z_0 + y_1z_1 + y_2z_2)\end{aligned}\right\} = 3 \ldots\ldots (2).$$

Now the tangent plane at the point $(x_0\, y_0\, z_0)$ is parallel to the plane containing the other two points $(x_1\, y_1\, z_1)$ $(x_2\, y_2\, z_2)$ and passing through the origin: hence equations (1) give us

$$\frac{x_0x_1}{a^2} + \frac{y_0y_1}{b^2} + \frac{z_0z_1}{c^2} = 0$$

$$\frac{x_0x_2}{a^2} + \frac{y_0y_2}{b^2} + \frac{z_0z_2}{c^2} = 0$$

$$\text{and}\quad \frac{x_0x_0}{a^2} + \frac{y_0y_0}{b^2} + \frac{z_0z_0}{c^2} = 1$$

Square these equations and add them, and

$$\left.\begin{aligned}&\frac{x_0^2}{a^2} + \frac{y_0^2}{b^2} + \frac{z_0^2}{c^2} \\ &+ \frac{2x_0y_0}{a^2b^2}(x_0y_0 + x_1y_1 + x_2y_2)\end{aligned}\right\} = 1;$$

$$\left.\begin{aligned}\therefore\ &\frac{x_0y_0}{a^2b^2}(x_0y_0 + x_1y_1 + x_2y_2) \\ &+ \frac{x_0z_0}{a^2c^2}(x_0z_0 + x_1z_1 + x_2z_2) \\ &+ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\end{aligned}\right\} = 0.$$

Similarly,

$$\frac{x_1 y_1}{a^2 b^2}(x_0 y_0 + x_1 y_1 + x_2 y_2) + \frac{x_1 z_1}{a^2 c^2}(\ldots\ldots) + \frac{y_1 z_1}{b^2 c^2}(\ldots\ldots) = 0,$$

and

$$\frac{x_2 y_2}{a^2 b^2}(x_0 y_0 + x_1 y_1 + x_2 y_2) + \frac{x_2 z_2}{a^2 c^2}(\ldots\ldots) + \frac{y_2 z_2}{b^2 c^2}(\ldots\ldots) = 0;$$

$$\therefore \frac{(x_0 y_0 + x_1 y_1 + x_2 y_2)^2}{a^2 b^2} + \frac{(x_0 z_0 + x_1 z_1 + x_2 z_2)^2}{a^2 c^2} + \frac{(y_0 z_0 + y_1 z_1 + y_2 z_2)^2}{b^2 c^2} = 0;$$

$$\therefore \left.\begin{aligned} x_0 y_0 + x_1 y_1 + x_2 y_2 &= 0 \\ x_0 z_0 + x_1 z_1 + x_2 z_2 &= 0 \\ y_0 z_0 + y_1 z_1 + y_2 z_2 &= 0 \end{aligned}\right\}$$

Therefore from equations (2), we have, for the locus of the intersection of the tangent planes, the ellipsoid

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 3.$$

Problem XXII.

Two points A, B in a surface of the second order are joined and produced to C: from any point P in the surface, draw AP meeting the plane of contact (of the enveloping cone whose vertex is C) in the point Q: the locus of the intersections of CP, BQ will be a surface of the second order.

Let C be the origin, CBA the axis of x: then the equation to the surface may assume the form

$$ax^2 + by^2 + cz^2 + 2a'zy + 2a''x + 2b''y + 2c''z + d = 0,$$

and CB, CA will be given by the equation

$$ax^2 + 2a''x + d = 0.$$

Let $CA = p$, $CB = q$, x, y, z the co-ordinates of P.

The equations to AQ are

$$x' - p = \frac{x-p}{z} z', \qquad y' = \frac{y}{z} z'.$$

Now the equation to a tangent plane to the surface at the point (xyz) is

$$(x'-x)(ax + a'') + (y'-y)(by + a'z + b'')$$
$$+ (z'-z)(cz + a'y + c'') = 0,$$

or $$x'(ax + a'') + y'(by + a'z + b'') + z'(cz + a'y + c'')$$
$$+ a''x + b''y + c''z + d = 0,$$

and if this passes through C, it is satisfied by $x' = 0$, $y' = 0$, $z' = 0$;

$$\therefore\ a''x' + b''y' + c''z' + d = 0$$

is the equation to the plane of contact.

Hence for the point $Q(x_1 y_1 z_1)$ we have

$$a''\left(p + \frac{x-p}{z} z_1\right) + b'' \frac{y}{z} z_1 + c'' z_1 + d = 0,$$

$$\text{and } z_1 = \frac{-(d + a''p)z}{a''x + b''y + c''z - a''p},$$

So $$y_1 = \frac{-(d + a''p)y}{a''x + b''y + c''z - a''p},$$

$$\text{and } x_1 - p = \frac{-(d + a''p)(x-p)}{a''x + b''y + c''z - a''p}.$$

The equations to BQ are

$$\left.\begin{aligned} x' - q &= \frac{x_1 - q}{z_1} z' \\ y' &= \frac{y_1}{z_1} z' \end{aligned}\right\} \dots\dots\dots\dots\dots\dots (2),$$

when x_1, y_1, z_1 have the values above found, and the equations to CP are

$$\left.\begin{aligned} x' &= \frac{x}{z}z' \\ y' &= \frac{y}{z}z' \end{aligned}\right\} \quad \ldots(3).$$

Substituting in equations (2), we have

$$-(x'-q)\,.\,(d+a''p)\frac{z}{z'} = (p-q)\,.\,(a''x+b''y+c''z-a''p)$$
$$-(d+a''p)\,.\,(x-p) \ldots\ldots\ldots\ldots\ldots\ldots(4),$$

and between equations (3), (4) and the equation to the surface we have to eliminate x, y, z.

Now

$$(q-x')\,.\,(d+a''p)\frac{z}{z'} = (p-q)\,.\left(\frac{a''x'}{z'}z + b''\frac{y'}{z'}z + c''z - a''p\right)$$
$$-(d+a''p)\frac{x'}{z'}z,$$

or $z\{(q-x')\,.\,(d+a''p) - (p-q)\,.\,(a''x'+b''y'+c''z')$
$$+(d+a''p)x'\} = -(p-q)a''pz',$$

$$\text{or } z = -\frac{a''p(p-q)}{A}z',$$

where $\quad A = q(d+a''p) - (p-q)\,.\,(a''x'+b''y'+c''z')$;

$$\therefore\ y = \frac{y'}{z'}z = -\frac{a''p(p-q)}{A}y',$$

$$x = \frac{x'}{z'}z = -\frac{a''p(p-q)}{A}x'.$$

Substituting these values in the equation to the surface, we have

$$\{a''p(p-q)\}^2\,.\,(ax'^2+by'^2+cz'^2+2a'z'y')$$
$$-2a''p(p-q)\,.\,(a''x'+b''y'+c''z')A + dA^2 = 0,$$

or we have for the locus of the intersections of CP, BQ the surface of the second order whose equation is

$$\begin{aligned}\{a''p(p-q)\}^2 . (ax'^2 + by'^2 + cz'^2 + 2a'x'y') \\ - 2a''p(p-q) . (a''x' + b''y' + c''z') \times \\ \{q(d+a''p) - (p-q) . (a''x' + b''y' + c''z')\} \\ + d\{q(d+a''p) - (p-q) . (a''x' + b''y' + c''z')\}^2 = 0.\end{aligned}$$

Problem 23 is discussed in Prof. Miller's *Crystallography*, Art. 28.

Problem XXIV.

A pencil of rays is incident parallel to the axis x *of an ellipsoid whose greatest and least axes are* a *and* c *respectively: find the nature and limits of the two curvilinear boundaries of the portion of the plane* xy *through which all the rays will pass, and show that if* e, e′, e″ *be the eccentricities of the principal sections in the planes* xy, xz, yz, *respectively, and* $\mu = \frac{1}{e}$, *the boundaries will be two ellipses whose semiaxes are* ae''^2, be''^2, *and* $\frac{cae}{be''}$; *but if* $\mu = \frac{1}{e'}$, *all the rays will pass through a portion of the arc of an ellipse, whose semiaxes are* ae′, be″, *included between the vertex and a double ordinate at a distance* $\frac{cae}{b}$ *from its centre.*

Let $\frac{X-x}{\cos\alpha} = \frac{Y-y}{\cos\beta} = \frac{Z-z}{\cos\gamma}$ be the equations to the refracted ray incident at the point (xyz) of the surface, ϕ the angle between this ray and the normal.

Then $$\cos\phi = \frac{-p\cos\alpha - q\cos\beta + \cos\gamma}{\sqrt{1+p^2+q^2}}.$$

Now

$$\frac{1+q^2}{1+p^2+q^2}=\mu^2\sin^2\phi=\mu^2\left\{1-\frac{(p\cos\alpha+q\cos\beta-\cos\gamma)^2}{1+p^2+q^2}\right\};$$

$$\therefore\ 1+q^2=\mu^2\{1+p^2+q^2-(p\cos\alpha+q\cos\beta-\cos\gamma)^2\}\ldots(1).$$

Now the equation to the normal plane being

$$Z-z=\frac{b^2}{c^2}\cdot\frac{z}{y}\cdot(Y-y),$$

we have $$\frac{\cos\gamma}{\cos\beta}=\frac{Z-z}{Y-y}=\frac{b^2}{c^2}\cdot\frac{z}{y}=-\frac{1}{q}.$$

Also $$\cos^2\alpha+\cos^2\beta+\cos^2\gamma=1,$$

$$\text{or}\quad\frac{\cos\alpha}{\cos\gamma}=\sqrt{1+q^2}\,.\cot\alpha.$$

Now from (1) we have

$$(\mu^2-1).(1+q^2)+\mu^2p^2=\mu^2(p\cos\alpha-\sqrt{1+q^2}\sin\alpha)^2.$$

Hence

$$\{(\mu^2-1)\sqrt{1+q^2}\cot\alpha+\mu^2p\}^2=\mu^2p^2+(\mu^2-1).(1+q^2);$$

$$\therefore\ \cot\alpha\sqrt{1+q^2}=\frac{-\mu^2p\pm\sqrt{\mu^2p^2+(\mu^2-1).(1+q^2)}}{\mu^2-1}.$$

Hence the equations to the refracted ray are

$$X-x=\frac{-\mu^2p\pm\sqrt{\mu^2p^2+(\mu^2-1).(1+q^2)}}{\mu^2-1}.(Z-z),$$

$$Y-y=-q(Z-z).$$

And when the ray meets the plane of xy, $Z=0$ and

$$X=x+\frac{\mu^2p\mp\sqrt{\mu^2p^2+(\mu^2-1).(1+q^2)}}{\mu^2-1},$$

$$Y=e'^2y.$$

Now $\quad p = -\frac{c^2}{a^2}\cdot\frac{x}{z}, \qquad q = -\frac{c^2}{b^2}\cdot\frac{y}{z}.$

Hence

$$X = x\left\{1 - \frac{\mu^2 c^2}{(\mu^2-1)a^2}\right\} - \frac{c^2}{\mu^2-1}\sqrt{\frac{\mu^2 x^2}{a^4} + \frac{\mu^2-1}{b^4}y^2 + \frac{\mu^2-1}{c^4}z^2}\ldots(2).$$

Now as x diminishes X decreases, and hence we shall obtain the interior boundary by making $x = 0$ and

$$\frac{y^2}{b^2} + \frac{z^2}{c^2} = 1.$$

This gives

$$X = -\frac{c^2}{\mu^2-1}\sqrt{(\mu^2-1)\frac{y^2}{b^4} + \frac{\mu^2-1}{c^2}\left(1 - \frac{y^2}{b^2}\right)}$$

$$= -\frac{c^2}{\sqrt{\mu^2-1}}\sqrt{\frac{y^2}{b^4} + \frac{\left(1 - \frac{y^2}{b^2}\right)}{c^2}}\ldots\ldots\ldots\ldots\ldots\ldots($$

$$= -\frac{c}{\sqrt{\mu^2-1}}\sqrt{1 - \left(1 - \frac{c^2}{b^2}\right)\cdot\frac{Y^2}{b^2 e''^4}};$$

$$\therefore\ X^2\cdot\frac{\mu^2-1}{c^2} + \frac{Y^2}{b^2 e''^2} = 1.$$

Now if $\mu = \frac{1}{-}$, then $\frac{c}{\sqrt{\mu^2-1}} = \frac{c}{\sqrt{\frac{1}{1-\frac{b^2}{a^2}} -}}$

$$= \frac{cae}{b}.$$

Hence the interior boundary is an ellipse whose semi-axes are be'' and $\frac{cae}{b}$. The exterior boundary will be found by putting $z = 0$ and $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ in equation (2).

Hence we hav

$$X = x\left\{1 - \frac{\mu^2 c^2}{(\mu^2-1)a^2}\right\} - \frac{c^2}{\mu^2-1}\sqrt{\frac{\mu^2}{a^2}\left(1-\frac{y^2}{b^2}\right) + \frac{\mu^2-1}{b^4}y^2} \ldots (4$$

Now when $\mu = \frac{1}{e}$ or $\frac{\mu^2}{\mu^2-1} = \frac{a^2}{b^2}$, this becomes

$$X = e''^2 x - \frac{c}{\mu^2-1}\sqrt{\frac{1-\frac{y^2}{b^2}}{a^2-b^2} + \frac{y}{b^2(a^2-b}}$$

$$= e''^2 x - \frac{c}{b\sqrt{\mu^2-}} - \ldots \sqrt{1-\frac{y^2}{b^2}} - \ldots \sqrt{\mu^2-1}$$

$$\frac{\left(X + \frac{c^2}{b\sqrt{\mu^2-1}}\right.}{a^2 e''^4} + \frac{Y^2}{e''^4 b^2} = 1.$$

Hence the exterior boundary is an ellipse whose semi-axes are $a\epsilon''^2$, $b\epsilon''^2$.

II. Again, in equation (3), let $\mu = \frac{1}{e'}$, and we have the equation

$$X^2 . \frac{\mu^2-1}{a^2} + \frac{X^2}{b^2 e''^4} = 1,$$

$$\text{or since } \frac{\mu^2-1}{a^2} = \frac{1}{a^2-c^2} = \frac{1}{a^2 e'^2},$$

the interior boundary is an ellipse whose semi-axes are $a\epsilon'$ and $b\epsilon''$.

In equation (4), let $\mu = \frac{1}{e'}$, then*

$$X = -\frac{c^2}{\mu^2-1}\sqrt{\frac{1-\frac{y^2}{b^2}}{a^2-c^2} + \frac{c^2 y^2}{b^4(a^2-c^2)}}$$

$$= -\frac{c^2}{\mu^2 - 1} \cdot \frac{1}{\sqrt{a^2 - c^2}} \sqrt{1 - \frac{y^2 e''^2}{b^2}},$$

$$\frac{X^2}{a^2 - c^2} + \frac{Y^2}{b^2 e''^2} = 1.$$

Hence the two boundaries merge into an ellipse whose semi-axes are

$$a\sqrt{1 - \frac{c^2}{a^2}} = ae' \text{ and } be''.$$

But y cannot be $> b$;

$\therefore\ Y$ .... $> e''^2 b$;

therefore, from the above equation,

$$X^2 \text{ cannot be } < (a^2 - c^2)\left(1 - \frac{e''^4}{e''^2}\right),$$

$$\text{.........}\quad < (a^2 - c^2) \cdot \frac{c^2}{b^2},$$

$$\text{.........}\quad < \frac{c^2 a^2 e'^2}{b^2};$$

$$\therefore\ X \text{ ... . } < \frac{cae'}{b}.$$

Also, y .... < 0;

$\therefore\ Y$ .... < 0;

$\therefore\ X$ .... $> ae'$.

Hence, all the rays pass through the portion of the arc of an ellipse, included between the vertex at the distance ae' from the centre, and a double ordinate at the distance $\frac{cae'}{b}$ from the centre.

MR GASKIN'S

AND

MR THURTELL'S PAPER,

JANUARY 9, 1840.

Problem I.

The advance of the hour-hand of a watch beyond the minute-hand is measured by $15\frac{2}{3}$ *of the minute divisions, and it is between nine and ten o'clock; find the exact time indicated by the watch.*

Let x = number of minutes past 9 o'clock.

Then $45 + \dfrac{x}{12}$ = number of divisions the hour-hand is past 9;

$$\therefore\ 45 + \frac{x}{12} - x = 15\tfrac{2}{3} = \frac{47}{3};$$

$$\therefore\ 12x - x = 540 - 188,$$

$$\text{or}\quad x = \frac{352}{11} = 32;$$

therefore the time is 28 minutes to 10.

Problem II.

Eliminate ϕ and ι from the equations

$$\mathrm{A} = \left(\frac{\sin^2\phi}{\mathrm{a}^2} + \frac{\cos^2\phi}{\mathrm{b}^2}\right)\cos^2\iota + \frac{\sin^2\iota}{\mathrm{c}^2} \ldots\ldots\ldots\ldots\ldots (1),$$

$$\mathrm{B} = \frac{\cos^2\phi}{\mathrm{a}^2} + \frac{\sin^2\phi}{\mathrm{b}^2} \ldots\ldots \qquad .(2),$$

and $$\mathrm{C} = \left(\frac{1}{\mathrm{b}^2} - \frac{1}{\mathrm{a}^2}\right)\sin\phi\cos\phi\cos\iota \ldots\ldots\ldots\ldots\ldots\ldots (3).$$

We have

$$A\sec^2\iota + B = \frac{1}{a^2} + \frac{1}{b^2} + \frac{\tan^2\iota}{c^2};$$

$$\therefore\ \sec^2\iota = \frac{\dfrac{1}{a^2} + \dfrac{1}{b^2} - \left(B + \dfrac{1}{c^2}\right)}{A - \dfrac{1}{c^2}}.$$

Again, $$A\sec^2\iota - \frac{\tan^2\iota}{c^2} = \frac{\sin^2\phi}{a^2} + \frac{\cos^2\phi}{b^2}$$

$$\text{and } B = \frac{\cos^2\phi}{a^2} + \frac{\sin^2\phi}{b^2}$$

$$\therefore\ B - A\sec^2\iota + \frac{\tan^2\iota}{c^2} = \left(\frac{1}{a^2} - \frac{1}{b^2}\right)\cos 2\phi.$$

Also $$\left(\frac{1}{b^2} - \frac{1}{a^2}\right)\sin 2\phi = 2C\sec\iota;$$

$$\left(\frac{1}{b^2} - \frac{1}{a^2}\right)^2 = 4C^2\sec^2\iota + \left(B - A\sec^2\iota + \frac{\tan^2\iota}{c^2}\right)^2$$

$$= \frac{4C^2\left(\dfrac{1}{a^2} + \dfrac{1}{b^2} - B - \dfrac{1}{c^2}\right)}{A - \dfrac{1}{c^2}} + \frac{\left(2B - \dfrac{1}{a^2} - \dfrac{1}{b^2}\right)^2}{\left(A - \dfrac{1}{c^2}\right)^2};$$

$$\therefore \left\{\left(A - \frac{1}{c^2}\right) \cdot \left(\frac{1}{b^2} - \frac{1}{a^2}\right)\right\}^2$$

$$= 4C^2 \left(A - \frac{1}{c^2}\right) \cdot \left(\frac{1}{a^2} + \frac{1}{b^2} - B - \frac{1}{c^2}\right) + \left(2B - \frac{1}{a^2} - \frac{1}{b^2}\right)^2.$$

Problem III.

A given quantity of mercury is taken out of the tube of a wheel-barometer; find the corresponding error of the index.

Let R, r be the radii of the large and small tubes respectively, and let the volume of mercury taken out $= \pi R^2 d$. Then since the pressure of the air, or the distance between the surfaces of the mercury in the tubes remains the same, the mercury will fall in the smaller tube and rise in the larger, so that the elevation and depression together $= d$. Suppose x, y these spaces; then $x + y = d$.

Also $$\pi r^2 x = \pi R^2 y,$$

and $$y = \frac{r^2}{R^2}(d - y),$$

$$y = \frac{dr^2}{R^2 + r^2}.$$

Hence the ball floating on the surface of the mercury in the larger tube will on the whole be depressed by $d - y$, and the wheel whose radius is ρ, around which a string connected with the ball passes, will revolve through an angle $\frac{d - y}{\rho}$, and the error of the index

$$\frac{d}{\rho} \cdot \frac{R^2}{R^2 + r^2}$$

$$\frac{d}{\rho} \cdot \frac{R^2}{R^2 + r^2} \cdot \frac{180}{\pi} \text{ degrees.}$$

PROBLEM IV.

An inextensible string binds tightly together two smooth cylinders whose radii are given; find the length of the string, and the ratio of the mutual pressure between the cylinders to the tension by which it is produced.

Fig. 26. Let R, r be the radii, $OP = a$ the distance between the centres of two circular sections, θ the inclination of LO to OS.

$$\text{Then } LM^2 = NR^2 = a^2 - (R - r)^2,$$

$$\text{and } \cos\theta = \frac{R - r}{a};$$

$$\therefore\ LkN = 2\pi R - 2R\theta = 2R\left\{\pi - \cos^{-1}\left(\frac{R - r}{a}\right)\right\},$$

$$\text{and } MVR = 2rLOS = 2r\cos^{-1}\left(\frac{R - r}{a}\right);$$

therefore whole length of the string

$$= 2\sqrt{a^2 - (R - r)^2} + 2\pi R - 2(R - r)\cos^{-1}\left(\frac{R - r}{a}\right).$$

Again, if R be the mutual pressure at S, T the tension of the string, then

$$2T\sin\theta = R,$$

$$\text{and } \frac{R}{T} = 2\sqrt{1 - \left(\frac{R - r}{a}\right)^2}.$$

Problem V.

If θ be the angle which the focal distance to any point of an ellipse makes with the tangent, and ϕ the angle between the lines drawn from that point to the extremities of the axis major, then $2 \tan\theta = e \tan\phi$.

Fig. 27. Let SY, HZ be perpendiculars from the foci, x, y the co-ordinates of P, a, b the semiaxes of the ellipse.

$$\text{Then } \tan STP = \frac{SM}{CM} = \frac{SY - HZ}{YP + PZ} = \frac{ex}{\;} \tan SPY,$$

$$\text{or } \tan\theta = -\frac{a}{ex} \cdot \frac{b^2 x}{a^2 y}.$$

$$\text{Again, } \tan\phi = -\tan(PAC + PaC)$$

$$= -\frac{\dfrac{y}{a+x} + \dfrac{y}{a-x}}{1 - \dfrac{y^2}{a^2 - x^2}}$$

$$= -\frac{y}{e^2} \cdot \frac{2ab^2}{a^4 y^2}$$

$$= -\frac{2b^2}{a e^2 y};$$

$$\therefore\ 2\tan\theta = e\tan\phi$$

Problem VI.

If $c^{n-2} y^2 = (x - a_1) . (x - a_2) \dots (x - a_n)$ be the equation to a curve, it cannot have maxima and minima ordinates for more than $\frac{n}{2} - 1$ values of x when n is even, nor for more than $\frac{n-1}{2}$ values, when n is odd.

When n is even. In this case x may have *any* negative value, and *any* positive value greater than a_n.

But in order that y may have real values, the second member of the above equation must be positive. Hence the positive values of $x < a_n$ must lie between $0a_1$, a_2a_3, $a_4a_5, \ldots a_{n-2}a_{n-1}$. The curve therefore, in this case, consists of two infinite hyperbolic branches, between which there are $\frac{n-2}{2}$ or $\frac{n}{2} - 1$ *loops*, as in Fig. 29.

Now evidently there can be no minimum ordinates, the condition for which requires the existence of ordinates on both sides of them; and since each loop gives a maximum ordinate, the number of corresponding abscissæ cannot exceed $\frac{n}{2} - 1$.

When n is odd. Here x cannot be negative, but the equation gives possible and increasing values of y, while $x > a_n$. In this case then, we have an infinite parabolic branch which can give no maximum or minimum ordinate; and $\frac{n-1}{2}$ loops, corresponding to values of x lying between $a_1 a_2$, $a_3 a_4, \ldots a_{n-2} a_{n-1}$, each of which gives a maximum ordinate, but no minimum, for the reason before assigned. Hence there cannot be maximum or minimum ordinates for more than $\frac{n-1}{2}$ values of x.

Problem VII.

Two imperfectly elastic spheres attract one another with forces varying as $\frac{1}{D^2}$: find the greatest separation of their centres after n impacts, their original distance being given.

Let d be the original distance between their centres,

R the sum of their radii,

$m_1\, m_2$ their masses,

$x_1\, x_2$ their distances from their common centre of gravity, which remains at rest, at the time t,

$v_1\, v_2$ their velocities,

f the mutual action of two bodies, each of a unit of mass, and separated at the unit of distance.

The equations of motion are

$$m_1\, d_t^2\, x_1 = -\frac{f m_1 m_2}{r^2}, \quad m_2\, d_t^2\, x_2 = -\frac{f m_1 m_2}{r^2} \quad \text{if } r = x_1 + x_2;$$

$$\therefore\ 2m_1\, d_t^2 x_1\, d_t x_1 + 2m_2\, d_t^2 x_2\, d_t x_2 = -2 f m_1 m_2 \frac{d_t r}{\ } ;$$

$$\therefore\ m_1 v_1^2 + m_2 v_2^2 = C + 2 f m_1 m_2 \quad \ldots\ldots(1)$$

$$= C + 2 f m_1 m_2$$

$$\therefore\ m_1 v_1^2 + m_2 v_2^2 = 2 f m_1 m_2 \left(\frac{1}{r} - \frac{1}{d}\right) \ldots\ldots(2):$$

Again, $m_1 v_1 = m_2 v_2$.

Hence $v_1^2 = \dfrac{2 f m_2^2}{m_1 + m_2}\left(\dfrac{1}{R} - \dfrac{1}{d}\right)$

and $v_2^2 = \dfrac{2 f m_1^2}{m_1 + m_2}\left(\dfrac{1}{R} - \dfrac{1}{d}\right)$,

are the velocities of the spheres when they strike: and if V_1, V_2 be their velocities after impact, λ their common elasticity, we have

$$\left.\begin{aligned} V_1 + V_2 &= \lambda\,(v_1 + v_2) \\ \text{and } m_1 V_1 &= m_2 V_2 \end{aligned}\right\};$$

$$k = \frac{mcaz + c^2 y}{c^2 - z^2}.$$

Now $$a^2 + k^2 = r^2;$$

$$\therefore \{mx(c^2 - z^2) + z(maz + cy)\}^2 + m^2(mcaz + c^2y)^2 = m^2r^2(c^2 - z^2)^2,$$

the equation to the surface generated.

Problem XI.

If r *be the radius of the small circle inscribed in a spherical triangle,* R *the radius of the circumscribed circle,* r_1, r_2, r_3 *the radii of the three circles, each of which touches one side of the triangle, and the other two sides produced, then*

$$\cot r_1 + \cot r_2 + \cot r_3 - \cot r = 2 \tan R.$$

Prove this, and deduce the corresponding expression for a plane triangle.

Fig. 28. Let a, b, c, A, B, C denote the sides and angles of the spherical triangle, $2S = a + b + c$. Then by trigonometry

$$(\cot r)^2 D = \sin^2 S,$$

$$\text{and } (\tan R)^2 D = 4 \sin^2 \frac{a}{2} \sin^2 \frac{b}{2} \sin^2 \frac{c}{2},$$

$$\text{where } D = \sin S \sin(S - a) \sin(S - b) \sin(S - c).$$

Let $OK = OH = OL = r_1$, the angles HCK, LBK being bisected.

$$\text{Then} \quad \left.\begin{aligned} \sin CK &= \tan r_1 \tan \frac{C}{2} \\ \text{and } \sin BK &= \tan r_1 \tan \frac{B}{2} \end{aligned}\right\}$$

$$\therefore\ \cos a = \sqrt{1 - \tan^2 r_1 \tan^2 \frac{C}{2}}\sqrt{1 - \tan^2 r_1 \tan^2 \frac{B}{2}}$$

$$- \tan^2 r_1 \tan \frac{B}{2} \tan \frac{C}{2},$$

$$\cos^2 a + 2 \cos a \tan^2 r_1 \tan \frac{B}{2} \tan \frac{C}{2} = 1 - \tan^2 r_1 \left(\tan^2 \frac{B}{2} + \tan^2 \frac{C}{2}\right);$$

$$\therefore\ \cot^2 r_1 \sin^2 a = \tan^2 \frac{B}{2} + \tan^2 \frac{C}{2} + 2 \tan \frac{B}{2} \tan \frac{C}{2} \cos a$$

$$= \frac{\sin^2 (S-a)}{D}\left\{\{\sin (S-b) + \sin (S-c)\}^2 \right.$$

$$\left. - 4 \sin (S-b) \sin (S-c) \sin^2 \frac{a}{2}\right\}$$

$$= \frac{4 \sin^2 \frac{a}{2} \cos^2 \frac{a}{2} \sin^2 (S-a)}{D};$$

$$\left.\begin{aligned} \therefore\ \cot r_1 \sqrt{D} &= \sin (S-a) \\ \text{similarly}\ \cot r_2 \sqrt{D} &= \sin (S-b) \\ \text{and}\ \cot r_3 \sqrt{D} &= \sin (S-c) \end{aligned}\right\};$$

$$\therefore\ (\cot r_1 + \cot r_2 + \cot r_3 - \cot r)\sqrt{D}$$

$$= \sin (S-a) + \sin (S-b) + \sin (S-c) - \sin S$$

$$= 2 \sin \frac{c}{2} \cos \frac{a-b}{2} - 2 \cos \frac{a+b}{2} \sin \frac{c}{2}$$

$$= 4 \sin \frac{a}{2} \sin \frac{b}{2} \sin \frac{c}{2}$$

$$= 2 \tan R \sqrt{D};$$

$$\therefore\ \cot r_1 + \cot r_2 + \cot r_3 - \cot r = 2 \tan R.$$

Again, if ρ be the radius of the sphere, we have

$$\rho\left(\frac{1}{r_1}+\frac{1}{r_2}+\frac{1}{r_3}-\frac{1}{r}\right)-\frac{1}{3}\cdot\left(\frac{r_1+r_2+r_3-r}{\rho}\right)+\&c.$$

$$=2\left\{\frac{R}{\rho}+\frac{1}{3}\cdot\left(\frac{R}{\rho}\right)^3-\&c.\right\};$$

$$\frac{1}{r_1}+\frac{1}{r_2}+\frac{1}{r_3}-\frac{1}{r}$$

$$\frac{1}{3}\cdot\left(\frac{r_1+r_2+r_3-r}{\rho^2}\right)+\&c.$$

$$+2\left\{\frac{R}{\rho^2}+\frac{1}{3\rho}\left(\frac{R}{\rho}\right)^3-\&c.\right\}.$$

Hence, making $\rho=\infty$, we have

$$\frac{1}{r_1}+\frac{1}{r_2}+\frac{1}{r_3}=\frac{1}{r}.$$

This property of the inscribed and escribed circles, together with many others, may be found in the *Correspondance Mathématique* par M. Quetelet, Tome v.

PROBLEM XII.

A hollow cube, filled with heavy fluid, is held with one diagonal vertical; find the centre of pressure of one of the lower faces.

Let the lowest point of the diagonal be the origin, and the edges meeting in that point axes of co-ordinates, the axis of z being perpendicular to the face we are considering, ρ the density of the fluid, p the pressure at the point (xyz), a a side of the cube. The equation for finding p is (Poisson's *Hydrostatique*, Art. 583)

$$dp=\rho\,(Xdx+Ydy+Zdz)$$

$$=\frac{g\rho}{\sqrt{3}}(dx+dy+dz),$$

$$\therefore\ p = \frac{g\rho}{\sqrt{3}}(x + y + z) + C.$$

Now when $x = 0$, $y = 0$, $z = 0$, $p = g\rho a\sqrt{3}$;

$$\therefore\ p = \frac{g\rho}{\sqrt{3}}(x + y + z + a\sqrt{3}),$$

or making $z = 0$, the pressure at the point (xy) of on of the lower faces

$$= \frac{g\rho}{\sqrt{3}}(x + y + a\sqrt{3});$$

therefore pressure on an element

$$= \frac{g\rho}{\sqrt{3}}(x + y + a\sqrt{3})\,\delta x\,\delta y, \text{ ultimately.}$$

Moment of this about $x = \dfrac{g\rho}{\sqrt{3}}(x + y + a\sqrt{3})\,y\,\delta x\,\delta y$,

.... $y = \dfrac{g\rho}{\sqrt{3}}(x + y + a\sqrt{3})\,x\,\delta x\,\delta y$.

Whole pressure $= \dfrac{g\rho}{\sqrt{3}}\,{}_0\!\int_x^a \left(xa + \dfrac{a^2}{2} + a^2\sqrt{3}\right)$

$$= g\rho a^3 . \frac{1 + \sqrt{3}}{\sqrt{3}}.$$

Whole moment about $x = \dfrac{g\rho}{\sqrt{3}}\,{}_0\!\int_x^a {}_0\!\int_y^a (xy + y^2 + ay\sqrt{3})$

$$= \frac{g\rho a^4}{12} . \frac{7 + 6\sqrt{3}}{\sqrt{3}}.$$

Whole moment about $y = \dfrac{g\rho a^4}{12} . \dfrac{7 + 6\sqrt{3}}{\sqrt{3}}$.

Hence if (x), (y) be the co-ordinates of the centre of pressure, we have

$$(x) = (y) = \frac{a}{12} \cdot \frac{7 + 6\sqrt{3}}{1 + \sqrt{3}}$$

$$= \frac{a}{24} \cdot (\sqrt{3} + 11).$$

PROBLEM XIII.

A body is placed within a hollow cube, and attracted to its angular points by equal forces varying as the distance; determine the motion.

Let x, y, z be the co-ordinates of the particle at time t from the commencement of the motion, the middle point of the diagonal or the position of rest of the particle being the origin, and axes parallel to three adjacent edges of the cube, $2a$ a side of the cube, α, β, γ the original values of x, y, z.

Then 4 of the centres of force situated in the face perpendicular to the axis of x, and on the positive side of the origin, each exert a force $\mu (a - x)$ to increase x: and the 4 situated on the other side to diminish x by a force $\mu (a + x)$;

$$\therefore\ d_t^2 x = 4\mu (a - x) - 4\mu (a + x),$$

$$\left.\begin{aligned} \text{or } d_t^2 x + 8\mu x &= 0 \\ \text{so } d_t^2 y + 8\mu y &= 0 \\ d_t^2 z + 8\mu z &= 0 \end{aligned}\right|$$

$$\therefore\ z d_t^2 x - x d_t^2 z = 0,$$

$$z d_t x - x d_t z = C = 0,$$

$$\text{and } z = Cx = \frac{\gamma}{a} x.$$

Hence the motion of the particles is in a straight line passing through the origin or position of rest, whose equations are

$$\frac{x}{\alpha} = \frac{y}{\beta} = \frac{z}{\gamma}.$$

Let r be the distance of the particle from the origin. Then

$$(d_t r)^2 = (d_t x)^2 + (d_t y)^2 + (d_t z)^2$$

$$= C - 8\mu r^2,$$

$$\text{and } d_t^2 r + 8\mu r = 0,$$

a linear equation of the second order, the solution of which is

$$r = A\cos\left(2\sqrt{2\mu}\,.\,t + B\right)$$

$$= \sqrt{\alpha^2 + \beta^2 + \gamma^2}\,.\cos 2\sqrt{2\mu}\,.\,t.$$

Now this expression undergoes every change of value while $2\sqrt{2}\,.\,t$ is increased by π, and the time of oscillation from rest to rest $= \dfrac{\pi}{2\sqrt{2\mu}}$, the amplitude of the oscillation being $\sqrt{\alpha^2 + \beta^2 + \gamma^2}$.

Problem XIV.

Shew that

$$\int_x \frac{\alpha + \beta\cos x}{(a + b\cos x)^m} = (-1)^{m-1}$$

$$\frac{2}{1\,.\,2\,.\,3\ldots(m-1)\,b}\, d_a^{m-1}\left(\frac{\alpha b - a\beta}{\sqrt{a^2 - b^2}}\tan^{-1}\sqrt{\frac{a-b}{a+b}}\tan\frac{x}{2}\right).$$

Deduce by this method $\displaystyle\int_x \frac{1}{(a + b\cos x)^2}$.

Also shew that

$$\int_x \frac{x^{m-1}}{(a+bx^n)^p} = \frac{(-1)^r}{(p-1).(p-2)\ldots(p-r)}\, d_a^r \int_x \frac{x^{m-1}}{(a+bx^n)^{p-r}}$$

$$\text{I.}\quad \int_a \int_x \frac{\alpha + \beta\cos x}{(a+b\cos x)^m} = -\int_x \frac{\alpha+\beta\cos x}{(a+b\cos x)^{m-1}} \cdot \frac{1}{m-1},$$

$$\int_a^2 \int_x \frac{\alpha+\beta\cos x}{(a+b\cos x)^m} = \frac{1}{(m-1).(m-2)} \int_x \frac{\alpha+\beta\cos x}{(a+b\cos x)^{m-2}},$$

&c. = &c.

$$\int_a^{m-1} \int_x \frac{\alpha+\beta\cos x}{(a+b\cos x)^m} = \frac{(-1)^{m-1}}{(m-1).(m-2)\ldots 3.2.1} \int_x \frac{\alpha+\beta\cos x}{a+b\cos x}.$$

$$\text{Now}\quad \int_x \frac{\alpha+\beta\cos x}{a+b\cos x} = \int_x \frac{\alpha + \frac{\beta}{b}(a+b\cos x - a)}{a+b\cos x}$$

$$= \frac{1}{b}\int_x \frac{\alpha b - a\beta}{a+b\cos x} + \frac{\beta}{b}x$$

$$= \frac{2}{b}\,\frac{\alpha b - a\beta}{\sqrt{a^2-b^2}} \tan^{-1}\left\{\sqrt{\frac{a-b}{a+b}}\tan\frac{x}{2}\right\} + \frac{\beta x}{b};$$

$$\therefore \int_x \frac{\alpha+\beta\cos x}{(a+b\cos x)^m}$$

$$= \frac{(-1)^{m-1}.2}{1.2.3\ldots(m-1)\,b}\, d_a^{m-1}\left\{\frac{\alpha b - a\beta}{\sqrt{a^2-b^2}}\tan^{-1}\left(\sqrt{\frac{a-b}{a+b}}\tan\frac{x}{2}\right)\right\}.$$

$$\text{Hence}\quad \int_x \frac{1}{(a+b\cos x)^2}$$

$$= -\frac{2}{b}\, d_a \left\{\frac{b}{\sqrt{a^2-b^2}}\tan^{-1}\left(\sqrt{\frac{a-b}{a+b}}\tan\frac{x}{2}\right)\right\}$$

$$= \frac{1}{a^2-b^2}\left\{\frac{2a}{\sqrt{a^2-b^2}}\tan^{-1}\left(\sqrt{\frac{a-b}{a+b}}\tan\frac{x}{2}\right)\right.$$

$$\left. -2\sqrt{a^2-b^2}\, d_a\left(\tan^{-1}\sqrt{\frac{a-b}{a+b}}\tan\frac{x}{2}\right)\right\}.$$

Now $d_a \tan^{-1}\left(\sqrt{\frac{a-b}{a+b}}\tan\frac{x}{2}\right)$

$$= \sqrt{\frac{a+b}{a-b}} \cdot \frac{b\tan\frac{x}{2}}{(a+b)^2 + (a^2-b^2)\tan^2\frac{x}{2}},$$

$$\therefore \int_x \frac{1}{(a+b\cos x)^2}$$

$$= \frac{1}{a^2-b^2}\left\{\frac{2a}{\sqrt{a^2-b^2}}\tan^{-1}\left(\sqrt{\frac{a-b}{a+b}}\tan\frac{x}{2}\right)\right.$$

$$\left. - \frac{2b\tan\frac{x}{2}\cos^2\frac{x}{2}}{(a+b)\cos^2\frac{x}{2} + (a-b)\sin^2\frac{a}{2}}\right\}$$

$$= \frac{1}{a^2-b^2}\left\{\frac{2a}{\sqrt{a^2-b^2}}\tan^{-1}\left(\sqrt{\frac{a-b}{a+b}}\tan\frac{x}{2}\right) - \frac{b\sin x}{a+b\cos x}\right\}.$$

Again,

$$\int_a \int_x \frac{x^{m-1}}{(a+bx^n)^p} = -\frac{1}{p-1}\int_x \frac{x^{m-1}}{(a+bx^n)^{p-1}}$$

$$\int_a^2 \int_x \frac{x^{m-1}}{(a+bx^n)^p} = \frac{1}{(p-1)\,.\,(p-2)}\int_x \frac{x^{m-1}}{(a+bx^n)^{p-2}}$$

&c. = &c.

$$\int_a^r \int_x \frac{x^{m-1}}{(a+bx^n)^p} = \frac{(-1)^r}{(p-1)(p-2)\ldots(p-r)}\int_x \frac{x^{m-1}}{(a+bx^n)^{p-r}}$$

$$\int_x \frac{x^{m-1}}{(a+bx^n)^p} = \frac{(-1)^r}{(p-1)(p-2)\ldots(p-r)}\, d_a^r \int_x \frac{x^{m-1}}{(a+bx^n)^{p-r}}.$$

PROB

A plane passing through the centre of an ellipsoid is inclined to the plane of xy *at an angle* ι, *and its trace on the plane of* xy *is inclined to the axis of* x *at an angle* ϕ; $2a'$, $2b'$ *are the axes of the elliptic section, and* α *the angle which its axis major makes with the trace of the cutting plane on the plane of* xy. *Prove that* $\tan 2\alpha = \dfrac{2C}{A-B}$,

$$\frac{1}{a'^2} = \frac{A+B-\sqrt{(A-B)^2+4C^2}}{2}, \quad \frac{1}{b'^2} = \frac{A+B+\sqrt{(A-B)^2+4C^2}}{2}.$$

where A, B, C *have the values given in Problem* (2).

The equations for changing the co-ordinates, referred to the principal axes of the ellipsoid, to those in the plane of the section, which we shall denote by x', y', are

$$x = x' \cos\phi + y' \cos\iota \sin\phi,$$
$$y = x' \sin\phi - y' \cos\iota \cos\phi,$$
$$z = y' \sin\iota.$$

These are Euler's formulæ, and being substituted in the equation to the ellipsoid, give for the equation to the section

$$\left(\frac{x' \cos\phi + y' \cos\iota \sin\phi}{a}\right)^2 + \left(\frac{x' \sin\phi - y' \cos\iota \cos\phi}{b}\right)^2 + \frac{y'^2 \sin^2\iota}{c^2} = 1,$$

or suppressing the accents,

$$Ay^2 + Bx^2 - 2Cxy = 1.$$

Now in order to get the axes of this ellipse, we must use the formulæ for changing the axes into principal ones,

$$\left.\begin{aligned} x &= x' \cos a - y' \sin a \\ y &= x' \sin a + y' \cos a \end{aligned}\right\},$$

and the equation becomes, suppressing the accents,

$$a_1 y^2 + \beta_1 x^2 = 1,$$

where

$$\left.\begin{aligned} \beta_1 &= A \sin^2 a + B \cos^2 a - 2C \sin a \cos a \\ a_1 &= A \cos^2 a + B \sin^2 a + 2C \sin a \cos a \end{aligned}\right\},$$

under the condition

$$(A - B) \sin a \cos a + C (\sin^2 a - \cos^2 a) = 0,$$

$$\text{or} \quad \tan 2a = \frac{2C}{A - B} \text{..................} (1).$$

Now a is the angle at which the new axis of x is inclined to the old one, or the inclination of the trace of the plane to the principal axis of the section.

Again, $\quad a_1 + \beta_1 = A + B,$

$$a_1 - \beta_1 = (A - B) \cos 2a + 2C \sin 2a$$

$$= \frac{(A - B)^2 + 4C^2}{\sqrt{(A - B)^2 + 4C^2}} = \sqrt{(A - B)^2 + 4C^2};$$

$$\therefore \; 2a_1 = A + B + \sqrt{(A - B)^2 + 4C^2},$$

$$2\beta_1 = A + B - \sqrt{(A - B)^2 + 4C^2},$$

$$\text{and} \quad \frac{1}{a'^2} = \beta_1 = \frac{A + B - \sqrt{(A - B)^2 + 4C^2}}{2}$$

$$\frac{1}{b'^2} = a_1 = \frac{A + B + \sqrt{(A - B)^2 + 4C^2}}{2}$$

Problem XVI.

Two semicircular self-luminous plates are placed with their diameters upon a plane, and with their planes perpendicular to the line which joins their centres A *and* B; *find the position of that point in the line* AB, *where the illumination of the plane upon which the semicircles are placed is greatest.*

Let a, b be the radii of the semicircles. Consider a point P in the line AB, whose distance from A is x. Let c be the distance between their centres. Consider an annulus of the semicircle A, whose radii are r and $r + \delta r$, the distance of every point of which from P is ultimately the same and equal to $\sqrt{x^2 + r^2}$.

Then the illumination at P from the annulus

$= K\pi r\delta r \cdot \dfrac{x}{(x^2 + r^2)^{\frac{3}{2}}}$, ultimately, (Griffin's *Optics*, Art. 12),

K being a constant depending on the nature of the source from whence the light proceeds.

Hence the illumination at P from the whole semicircle

$$= K\pi x \int_0^a \frac{r}{(x^2 + r^2)^{\frac{3}{2}}}$$

$$= K\pi - \frac{K\pi x}{\sqrt{a^2 + x^2}}.$$

Similarly the illumination from the other semicircle

$$= K\pi - \frac{K\pi (c - x)}{\sqrt{b^2 + (c - x)^2}};$$

therefore the whole illumination at P

$$= 2K\pi - K\pi \left\{\frac{x}{\sqrt{a^2 + x^2}} + \frac{c - x}{\sqrt{b^2 + (c - x)^2}}\right\},$$

and this being a maximum by the variation of x, we have

$$\frac{x}{\sqrt{a^2+x^2}} + \frac{c-x}{\sqrt{b^2+(c-x)^2}} = \text{minimum};$$

$$\therefore \frac{a^2}{(a^2+x^2)^{\frac{3}{2}}} = \frac{b^2}{\{b^2+(c-x)^2\}^{\frac{3}{2}}},$$

whence x may be found by the solution of a quadratic.

Cor. If $a = b$, we have

$$x = c - x,$$

$$\text{or} \quad x = \frac{c}{2},$$

or the point of maximum illumination is at the middle point between the centres, as might have been anticipated.

Problem XVII.

In a general declining dial the style is bent in its own plane through a given angle α; determine when the angular motion of the sun's shadow on the dial-plate is a maximum.

Fig. 54. Let OP be the direction of the earth's axis, POT the plane of the style which is perpendicular to the dial-plate NOT, S the sun when on the meridian, OQ the position of the edge of the style, $PT = h$, $PQ = \alpha$, $SP = \frac{\pi}{2} - \delta$ the co-declination of the sun, OL the 12 o'clock hour-line, OH the hour-line t hours after 12, $TPL = \phi$. Then $TPH = \phi - 15t$, and

$$\tan TH = \sin h \tan(\phi - 15t).$$

Now $d_t(TH)$ is a maximum when

$$\frac{1}{\cos^2(\phi - 15t) + \sin^2 h \sin^2(\phi - 15t)}$$

is greatest, or when $\phi - 15t = \frac{\pi}{2}$, or $\tan(\pi - 15t) = \cot\phi$. This, when ϕ is known, is an equation for determining on what hour of a given day the maximum (which does not vary in value from day to day) occurs.

Now we may determine ϕ from the following series of equations:

$$\cos PN = \cot N \cot \psi, \quad \text{if } TPN = \psi,$$

$$\cos SQ = \cos a \sin \delta - \sin a \cos \delta \cos \psi,$$

$$\sin Q = \sin \psi \cdot \frac{\cos \delta}{\sin SQ},$$

$$\sin (h - a) = \tan TL \cot Q,$$

$$\sin h = \tan TL \cot TPL.$$

Making use of these equations, we find that

$$\text{an}\,\phi = \frac{\sin \psi \sin (h - a)}{\sin h} \times$$

$$\frac{1}{\sqrt{\sec^2 \delta - (\cos a \tan \delta - \sin a \cos \psi)^2 - \sin^2 \psi}}$$

MR STEVENTON'S PAPER,

JANUARY 8, 1841.

PROBLEM I.

Having given $\tan\phi = \dfrac{\sin\theta\cos\theta'}{\sin\theta' + \cos\theta}$, *shew that*

$$\tan\frac{\phi}{2} = \tan\frac{\theta}{2}\tan\left(45^0 - \frac{\theta'}{2}\right);$$

shew also that the equation

$$a^2b^4(x'-x)^2 + a^4b^2(y'-y)^2 + (b^2x^2 + a^2y^2 - a^2b^2)\,.\,(b^2x'^2 + a^2y'^2 - a^2b^2) = 0$$

is equivalent to the two

$$a^2b^2 - a^2yy' - b^2xx' = 0, \qquad \text{and } xy' - yx' = 0.$$

I. We have

$$\frac{1-\tan^2\dfrac{\phi}{2}}{1+\tan^2\dfrac{\phi}{2}} = \cos\phi = \frac{\sin\theta' + \cos\theta}{\sqrt{(\sin\theta' + \cos\theta)^2 + \sin^2\theta\cos^2\theta'}}$$

$$= \frac{\sin\theta' + \cos\theta}{\sqrt{1 + \cos^2\theta\sin^2\theta' + 2\sin\theta'\cos\theta}}$$

$$= \frac{\sin\theta' + \cos\theta}{1 + \sin\theta'\cos\theta};$$

$$\therefore\ \tan^2\frac{\phi}{2} = \frac{(1-\cos\theta)\,.\,(1-\sin\theta')}{(1+\cos\theta)\,.\,(1+\sin\theta')}$$

$$= \frac{\sin^2\dfrac{\theta}{2}\,.\,\sin^2\left(45^0 - \dfrac{\theta'}{2}\right)}{\cos^2\dfrac{\theta}{2}\,.\,\cos^2\left(45^0 - \dfrac{\theta'}{2}\right)};$$

$$\therefore \tan\frac{\phi}{2} = \tan\frac{\theta}{2}\tan\left(45^\circ - \frac{\theta'}{2}\right).$$

II. The expression given when expanded is the same as

$$a^2b^4(x_1^2 - 2x_1x + x^2) + a^4b^2(y_1^2 - 2yy_1 + y^2)$$
$$+ b^4x^2x_1^2 + b^2a^2x^2y_1^2 - a^2b^4x^2 + a^2b^2y^2x_1^2 + a^4y^2y_1^2$$
$$- a^4y^2b^2 - a^2b^4x_1^2 - a^4b^2y_1^2 + a^4b^4 = 0,$$

or $(b^2xx_1 + a^2yy_1 - a^2b^2)^2 + a^2b^2(xy_1 - yx_1)^2 = 0.$

Hence $$\left.\begin{array}{r} b^2xx_1 + a^2yy_1 - a^2b^2 = 0 \\ xy_1 - yx_1 = 0 \end{array}\right\}.$$

Problem II.

What plane regular rectilineal figure is that in which a similar rectilinear figure can be inscribed whose area shall be $\left(\frac{1}{n}\right)^{\text{th}}$ *of the area of the former? Ex.* n = 2, n = 4.

Fig. 30. Let AB, CE be homologous sides of the figures.

Then $$n = \frac{AB^2}{CE^2}, \quad \text{or } AB = \sqrt{n}\,.\,CE,$$

$$\text{and } CE = 2CK = 2CB\sin ABO$$
$$= AB\cos\frac{AOB}{2}.$$

Now if m be the number of sides of the figures,

$$mAOB = 2\pi;$$

$$\therefore\ CE = AB\cos\frac{\pi}{m}.$$

Hence $$1 = \sqrt{n} \cdot \cos\frac{\pi}{m},$$

which gives m when n is given.

Let $n = 2$. Then $\cos\frac{\pi}{m} = \frac{1}{\sqrt{2}} = \cos\frac{\pi}{4}$,

or $m = 4$, and the figure a square.

When $n = 4$, $\cos\frac{\pi}{m} = \frac{1}{2} = \cos\frac{\pi}{3}$,

or $m = 3$, and the figure is an equilateral triangle.

Problem III.

If from one of the angles of a rectangle a perpendicular be drawn to its diagonal, and from the point of their intersection lines be drawn perpendicular to the sides which contain the opposite angle; shew that if p *and* p′ *be the lengths of the perpendiculars last drawn, and* d *the diagonal of the rectangle,* $p^{\frac{2}{3}} + p'^{\frac{2}{3}} = d^{\frac{2}{3}}$.

Fig. 31. Let CD be perpendicular to AB; DF, DE perpendicular to BG, AG.

$$AB = d, \qquad DF = p, \qquad DE = p'.$$

Then the triangles ACD, BCD, ADE, BDF are all similar;

$$\left.\begin{aligned} \therefore\ \frac{AD}{AC} &= \frac{p'}{AD}, & \frac{DB}{CB} &= \frac{p}{DB} \\ \text{and } \frac{AC}{d} &= \frac{p'}{AD}, & \frac{BC}{d} &= \frac{p}{DB} \end{aligned}\right\},$$

$$\frac{AD}{d} = \left(\frac{p'}{AD}\right)^2, \qquad \frac{DB}{d} = \left(\frac{p}{DB}\right)^2;$$

$$\therefore\ d = AD + DB$$

$$= p'^{\frac{2}{3}} d^{\frac{1}{3}} + p^{\frac{2}{3}} d^{\frac{1}{3}};$$

$$\therefore\ d^{\frac{2}{3}} = p^{\frac{2}{3}} + p'^{\frac{2}{3}}.$$

Problem IV.

A hollow vessel in the form of a tetrahedron is filled with a given weight w *of fluid, and is placed in an inverted position on a horizontal plane; shew that the fluid will escape from under the vessel unless the weight of the vessel be at least equal to twice* w.

Fig. 32. Let CAB be a side of the tetrahedron, CY vertical, PM horizontal, ρ the density of the fluid. Then in order that the tetrahedron may be at rest, its weight must be at least equal to the vertical part of the pressure of the fluid on its surface.

Let $CP = x$, δ the inclination of the face CAB to the base.

The pressure at any point P in a direction perpendicular to CAB is the same as at M in the same horizontal plane. Hence the vertical pressure on an element at P

$$= g\rho x^2 \cdot \frac{AB}{CD} \cdot \cos\delta \sin\delta \cdot \delta x,$$

and the whole vertical pressure on CAB

$$= \tfrac{2}{3} g\rho CY \cdot \text{area } AYB.$$

And similarly for the vertical pressures on the other faces. Hence the whole vertical pressure on the faces $= \frac{2}{3}$ weight of a prism of the fluid on ABE, and having CY as its altitude, and w is $\frac{1}{3}$ the weight of such a prism; hence the weight of the tetrahedron must be at least twice w.

Problem V.

A cone rests with its base upon the vertex of a given paraboloid; find the greatest ratio which the height of the cone can bear to the length of the latus rectum of the paraboloid, while the equilibrium remains stable.

Fig. 38. Let the cone be slightly displaced in a vertical plane, so that P is the point of contact, PL parallel to its axis which meets the axis of the paraboloid in K, A being its vertex; PR vertical meeting the axis of the cone, whose centre of gravity is G, in R. PL is normal at P, being perpendicular to the base or tangent. As P moves up to A, K approaches A, and L ultimately becomes the centre of curvature at A.

Now the equilibrium is stable so long as the perpendicular from P on the vertical through G falls to the left of P, since the tendency of the cone will then be to return to its position of rest, and this will be the case if $aG < aR$ or $< PR$ ultimately.

$$< KL \quad < \text{radius of curvature at } A.$$

Now $aG = \frac{1}{4}$ height of cone, and radius of curvature $= \frac{1}{2}$ latus rectum. The equilibrium is stable therefore so long as height of cone < 2 latus rectum. The greatest ratio therefore which the height of the cone can bear to the latus rectum of the paraboloid is 2, at which point the equilibrium is neutral.

Problem VI.

A hollow square and a hollow equilateral wedge, each r *deep, have the same number* n *of men in each of their fronts, also the number of men in the innermost ranks of the square is equal to* p *times the corresponding number in the wedge; shew that*

$$r(9p-8) = n(3p-4) + 6p - 4.$$

In the equilateral wedge the men must be placed so that any 3 adjacent ones not in the same line may form an equilateral triangle, and the square wedge must be made up of small squares.

In the former case, as we proceed inwards, the ranks will diminish 3 in number at each step, and 4 in the latter. Hence the interior ranks will be respectively an equilateral triangle having $n - 3(r - 1)$ men in a side, and a square containing in each side $n - 2(r - 1)$ men. Hence

$$4\{n - 2(r - 1) - 1\} = 3p\{n - 3(r - 1) - 1\};$$

$$\therefore\ r(9p - 8) = n(3p - 4) + 6p - 4.$$

Problem VII.

A small pencil of rays is transmitted through a prism in a plane perpendicular to its axis; find the focus of emergent rays in the primary plane, taking account of the thickness of the prism, and deduce from the result the expression for the place of the focus when a pencil is transmitted through a plate which is bounded by parallel plane surfaces.

Fig. 34. Let $QRST$ be the course of a ray in the plane of the paper, RK normal at R, Q_1, q_1 the foci after refraction at the first and second surfaces, ϕ, ϕ' the angle of incidence and refraction at the first surface, ψ', ψ at the second.

$$QR = u_1, \qquad Sq_1 = v_1, \qquad RK = t, \qquad KAR = \iota.$$

Then for refraction at the first surface (Griffin's *Optics*, Art. 50.)

$$RQ_1 = \frac{\mu \cos^2\phi'}{\cos^2\phi} u_1,$$

and if the course of the pencil be reversed at the second surface,

$$SQ_1 = \frac{\mu \cos^2 \psi'}{\cos^2 \psi} v_1;$$

$$\text{and } SQ_1 = RQ_1 + SR = RQ_1 + \frac{t \cos \iota}{\sin (90 - \iota - \phi')}.$$

$$\therefore\ SQ_1 = \frac{\mu \cos^2 \phi'}{\cos^2 \phi} u + \frac{t \cos \iota}{\cos (\iota + \phi')} = \frac{\mu \cos^2 \psi'}{\cos^2 \psi} v_1,$$

and this equation gives v_1.

Now let $\quad \psi' = \phi', \qquad \psi = \phi, \qquad \iota = 0.$

$$\text{Then} \qquad v_1 = u + \frac{}{\cos \phi'} \cdot \frac{\cos^2 \phi}{\mu \cos^2 \phi'} = u + \frac{t \cos^2 \phi}{\mu \cos^3 \phi'},$$

which is the expression for finding the focus in the case of a plate bounded by parallel surfaces.

Problem VIII.

If a prismatic diving-bell of given volume be sunk to a given depth in water, find the volume of air (at its natural density) which will be required to be forced into the bell, in order that $\left(\frac{1}{m}\right)^{\text{th}}$ *of its volume may be free from water.*

Let d be the depth of the upper surface of the vessel whose altitude is a, and the area of whose base is A, h the height of a column of water whose weight equals the pressure of the air, V the volume of air forced into the bell, ρ the density of water. Then pressure of the water at its surface inside the bell

$$= g\rho \left(h + d + \frac{a}{m}\right).$$

and pressure of the confined air

$$=g\rho h.\frac{Aa+V}{\frac{1}{m}.Aa}.$$

Now these must be equal to each other;

$$\therefore\ Aa\left(h+d+\frac{a}{m}\right)=mh\,(Aa+V),$$

which gives V.

Problem IX.

Shew that the curve whose equation is $\frac{y}{r}=sin\frac{a}{r}$, *where* $y^2-2rx+x^2=0$ *intersects the axis of* x *in an infinite number of points; also that the curve*

$$ye^{\frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{2x-1}{\sqrt{3}}\right)}=c\sqrt{x^2-x+1}$$

is one of those denoted by the equations

$$log\left(\frac{y}{c}\right)=\int_x\frac{1}{x-\phi(x)},\qquad and\ \phi^2(x)=x.$$

I. When the substitutions are made, the equation to the curve becomes

$$\frac{2xy}{x^2+y^2}=\sin\frac{2ax}{x^2+y^2},$$

and when $y=0$, we have

$$\sin\frac{2a}{x}=0=\sin m\pi,$$

$$\text{or } x=\frac{2a}{m\pi},$$

where m is any integer whatever. Hence there are an indefinite number of points in which the curve cuts the axis of x.

II. The simplest solution of $\phi^3(x) = x$, $\phi(x)$ being a periodic function, is (Babbage's *Examples*, Ex. 1)

$$\phi(x) = \frac{1}{1-x}.$$

For
$$\phi^2(x) = \frac{1}{1 - \frac{1}{1-x}} = \frac{x-1}{x},$$

$$\phi^3(x) = \frac{\frac{1}{1-x} - 1}{\frac{1}{1-x}}.$$

Hence
$$\log\left(\frac{y}{c}\right) = \int_x \frac{1}{x - \frac{1}{1-x}} = \int_x \frac{-1+x}{x^2 - x + 1}$$

$$= \log_e \sqrt{x^2 - x + 1} - \frac{1}{\sqrt{3}} \tan^{-1}\left(\frac{2x-1}{\sqrt{3}}\right);$$

$$\therefore \frac{1}{\sqrt{3}} \tan^{-1}\left(\frac{2x-1}{\sqrt{3}}\right) = \log_e \frac{c\sqrt{x^2 - x + 1}}{y},$$

$$\text{or } y e^{\frac{1}{\sqrt{3}} \tan^{-1}\left(\frac{2x-1}{\sqrt{3}}\right)} = c\sqrt{x^2 - x + 1}.$$

Problem X.

If two bodies be projected from the same point, at the same instant, with velocities v *and* v_1, *and in directions* a *and* a_1, *shew that the time which elapses between their transits through the other point which is common to both their paths is equal to* $\frac{2}{g} \cdot \frac{v v_1 \sin(a \sim a_1)}{v_1 \cos a_1 + v \cos a}$.

The equations to the paths of the projectiles are

$$\left.\begin{aligned} y &= x\tan\alpha - \frac{gx^2}{2v^2\cos^2\alpha} \\ y &= x\tan\alpha_1 - \frac{gx^2}{2v_1^2\cos^2\alpha_1} \end{aligned}\right\}.$$

And at their point of intersection

$$\tan\alpha - \frac{gx}{2v^2\cos^2\alpha} = \tan\alpha_1 - \frac{gx}{2v_1^2\cos^2\alpha_1},$$

$$\text{and } x = \frac{2}{g}\cdot\frac{(vv_1)^2\cos\alpha\cos\alpha_1\sin(\alpha \sim \alpha_1)}{v_1^2\cos^2\alpha_1 \sim v^2\cos^2\alpha}.$$

Also since the horizontal velocity is constant in both cases, the time elapsed

$$= \frac{x}{v\cos\alpha} \sim \frac{x}{v_1\cos\alpha_1}$$

$$= \frac{x(v_1\cos\alpha_1 \sim v\cos\alpha)}{vv_1\cos\alpha\cos\alpha_1}$$

$$= \frac{2}{g}\cdot\frac{vv_1\sin(\alpha \sim \alpha_1)}{v_1\cos\alpha_1 + v\cos\alpha}.$$

Problem XI.

A certain substance, whose volume is v *and density* ρ, *is enclosed in a thin expansible film and sunk in water. The substance undergoes a change (in proportion to the time) such that each portion of it swells into* m *times its original volume; shew that the substance will begin to rise in the water after* $\frac{v(\rho - 1)}{n(m - 1)}$ *seconds, if* n *be the portion of the substance changed in one second, and unity represent the density of water. Find also the portion of the substance which remains unchanged at that instant.*

Since n is the portion changed in 1 second,

nt .... in t seconds,

and mnt = volume of the portion changed;

$\therefore$ $mnt + v - nt$ = volume of the substance at the end of time t, which is also the mass of the fluid displaced by it; therefore when it is just on the point of rising,

$$mnt + v - nt = \rho v,$$

$$\text{or } t = \frac{v(\rho - 1)}{n(m-1)}.$$

And the portion of the substance which is unchanged

$$= v - nt = v - \frac{v(\rho - 1)}{m-1}$$

$$= v\left(\frac{m-\rho}{m-1}\right).$$

Problem XII.

If the density of a straight rod AB *vary as* (*dist.*)^n *from one end* A, *and* k, k$'$ *be the radii of gyration of the rod round* A *and* B *respectively, shew that* k^2 *is always some exact multiple of* k$'^2$, *whatever be the integral value of* n; *find also the value of* n *so that* k *may be equal to* 6k$'$.

Let P be a point in the rod at the distance r from A, δr an elementary portion of the rod. Then

moment of inertia of this element about $A = \delta r . \mu r^n . r^2$,

.... $B = \delta r . \mu r^n . (a - r)^2$,

if a = length of the rod;

therefore moment of inertia of the whole rod about A

$$= \mu\,{}^{0}\!\!\int_r^{a} r^{n+2} = \frac{\mu a^{n+3}}{n+3},$$

.... about B

$$= \mu\,{}^{0}\!\!\int_r^{a} (a^2 r^n - 2a r^{n+1} + r^{n+2})$$

$$= \mu a^{n+3} \left(\frac{1}{n+1} + \frac{1}{n+3} - \frac{2}{n+2}\right)$$

$$= \frac{2\mu a^{n+3}}{(n+1).(n+2).(n+3)}$$

Also mass of rod $= \dfrac{\mu a^{n+1}}{n+1}$;

$$\therefore\ k^2 = a^2 \frac{n+1}{n+3}, \qquad k'^2 = \frac{2a^2}{(n+2).(n+3)},$$

$$\frac{k^2}{k'^2} = \frac{(n+1).(n+2)}{2},$$

and $(n+1).(n+2)$ is always divisible by 2.

Let $\quad k = 6k'$.

Then $\quad 72 = (n+1).(n+2) = n^2 + 3n + 2$;

$$\therefore\ n^2 + 3n = 70,$$

$$n = -\frac{3}{2} \pm \sqrt{\frac{9}{4} + 70}$$

$$= 7, \quad \text{or} \quad -10.$$

Problem XIII.

A conical surface, whose vertex is A, is cut by a plane which is parallel to one of the slant sides AC, and the part of the surface which contains the vertex is developed into a plane, the surface being again cut

along AC; *shew that the polar equation to the curved boundary of the plane figure, measuring from* A, *is* $r = \frac{2\,AB}{1 + \cos(\theta \operatorname{cosec} \alpha)}$, *where* 2α *is the vertical angle of the cone, and* B *the point of intersection of the cutting plane with the slant side which is opposite to* AC.

Fig. 35. The plane into which the cone can be developed is that perpendicular to the axis of the cone. Let BP be the parabolic section parallel to the side AC of the cone, $AB = d$. Then its equation is $y^2 = 4d \sin^2 \alpha x$, and the equation to its projection on the above plane is $y^2 = 4d \sin \alpha x$.

Hence the vertex of the cone is the focus of the projection, the polar equation to which is therefore

$$\rho = \frac{2d \sin \alpha}{1 + \cos \phi}.$$

Now let $AP = r$ and θ the angle which AP makes with AB when the cone is developed. Then US is a portion of a circular arc of which AS is radius, subtending an angle θ at the centre A;

$$\therefore\ \phi . SO = \theta . AS;$$

$$\left.\begin{aligned} \therefore\ \phi &= \theta \operatorname{cosec} \alpha \\ \text{and } \rho &= r \sin \alpha \end{aligned}\right\};$$

$$\therefore\ r \sin \alpha = \frac{2d \sin \alpha}{1 + \cos(\theta \operatorname{cosec} \alpha)},$$

$$\text{or } r = \frac{2\,AB}{1 + \cos(\theta \operatorname{cosec} \alpha)}$$

is the equation to BP when the cone is developed.

Problem XIV.

If the condition that an equation of the second order may represent a parabola be introduced into the equation

$$(a^2\beta^2+b^2\alpha^2-a^2b^2).(a^2\beta_1^2+b^2\alpha_1^2-a^2b^2).(a^2y^2+b^2x^2-a^2b^2)$$
$$+a^4b^4\{y(\alpha_1-\alpha)-x(\beta_1-\beta)+\alpha\beta_1-\alpha_1\beta\}^2=0,$$

shew that the equation will be reduced to the form

$$\frac{a}{b}\sqrt{\beta\beta_1}\,.\,y+\frac{b}{a}\sqrt{\alpha\alpha_1}\,.\,x=\pm ab.$$

Let A, B, C be the coefficients of y^2, xy, x^2 respectively in the above equation, and

$$\phi=(a^2\beta^2+b^2\alpha^2-a^2b^2).(a^2\beta_1^2+b^2\alpha_1^2-a^2b^2).$$

Then, introducing the condition $4AC-B^2=0$, we have

$$\phi=-a^2b^2\{a^2(\beta_1-\beta)^2+b^2(\alpha_1-\alpha)^2\},$$

$$\text{or}\quad \left.\begin{array}{r}(a^2\beta^2+b^2\alpha^2-a^2b^2).(a^2\beta_1^2+b^2\alpha_1^2-a^2b^2)\\ +a^2b^2\{a^2(\beta_1-\beta)^2+b^2(\alpha_1-\alpha)^2\}\end{array}\right\}=0,$$

and this is equivalent to the equations

$$a^2b^2-a^2\beta\beta_1-b^2\alpha\alpha_1=0,$$
$$\alpha\beta_1=\alpha_1\beta.$$

Now
$$A=-a^4b^2(\beta_1-\beta)^2,$$
$$B=-2a^4b^4(\alpha_1-\alpha).(\beta_1-\beta),$$
$$C=-b^6a^2(\alpha_1-\alpha)^2.$$

Hence the equation becomes

$$\{a^2(\beta_1-\beta)y+b^2(\alpha_1-\alpha)x\}^2=-\phi$$
$$=-(a^2\beta^2+b^2\alpha^2-a^2b^2).(a^2\beta_1^2+b^2\alpha_1^2-a^2b^2)$$
$$=\{a^2\beta(\beta-\beta_1)+b^2\alpha(\alpha-\alpha_1)\}\times$$
$$\{a^2\beta_1(\beta-\beta_1)+b^2\alpha_1(\alpha-\alpha_1)\}$$
$$(\text{since }\alpha\beta_1=\alpha_1\beta)$$
$$=\{a^2\sqrt{\beta\beta_1}(\beta-\beta_1)+b^2\sqrt{\alpha\alpha_1}(\alpha-\alpha_1)\}^2;$$

$$\therefore\ a^2(\beta-\beta_1)y+b^2(\alpha-\alpha_1)x$$

$$=\pm\left\{a^2\sqrt{\beta\beta_1}\,(\beta-\beta_1)+b^2\sqrt{\alpha\alpha_1}\,(\alpha-\alpha_1)\right\},$$

$$\text{or since } \frac{\alpha-\alpha_1}{\beta-\beta_1}=\frac{\alpha}{\beta},$$

$$a^2\beta y+b^2\alpha x=\pm\left(a^2\beta\sqrt{\beta\beta_1}+b^2\alpha\sqrt{\alpha\alpha_1}\right),$$

$$\text{or}\quad a^2y\sqrt{\beta\beta_1}+b^2x\frac{\alpha}{\beta}\sqrt{\beta\beta_1}=\pm\left(a^2\beta\beta_1+b^2\frac{\alpha}{\beta}\right.$$

$$\text{or}\quad a^2y\sqrt{\beta\beta_1}+b^2x\sqrt{\alpha\alpha_1}=\pm\left(a^2\beta\beta_1+b^2\alpha\alpha_1\right)$$

$$=\pm a^2b^2;$$

Problem XV.

Two particles P *and* P′ *whose masses are* m *and* m′ *are connected by a straight rod, and, being constrained to move in two straight grooves* AP, AP′, *which are inclined to the horizon at given angles, make small oscillations in a vertical plane; shew that the length of the isochronous pendulum is* $\dfrac{\mathrm{m}\,.\,\mathrm{OP}^2+\mathrm{m}'\,.\,\mathrm{OP}'^2}{(\mathrm{m}+\mathrm{m}')\,\mathrm{GH}}$, *where* OP, OP′ *are perpendiculars to* AP, AP′, *and* GH *is the depth of the centre of gravity of* P *and* P′ *below the horizontal plane drawn through* A, *when the system is in equilibrium.*

Fig. 36. Let PP' be the position of equilibrium of the rod, pp' its position at time t, θ, $\theta-\delta\theta$ the angles which the rod makes with the horizon in the two positions, x, y, x_1, y_1 the horizontal and vertical co-ordinates of the extremities of the rod, T its tension, R, R' the

reactions of the grooves at p, p'; c the length of the rod, α, α_1 the inclinations of the grooves to the horizon.

The equations of motion are

$$m d_t^2 x = R \sin\alpha - T\cos(\theta - \delta\theta),$$

$$m d_t^2 y = -R\cos\alpha - T\sin(\theta - \delta\theta) + mg,$$

$$m' d_t^2 x_1 = R'\sin\alpha_1 - T\cos(\theta - \delta\theta),$$

$$m' d_t^2 y_1 = -R'\cos\alpha_1 + T\sin(\theta - \delta\theta) + m'g.$$

Eliminating T, R and R', we have

$$\frac{m(\cos\alpha\, d_t^2 x + \sin\alpha\, d_t^2 y - g\sin\alpha)}{\sin\alpha\sin(\theta - \delta\theta) + \cos\alpha\cos(\theta - \delta\theta)}$$

$$= \frac{m'(\cos\alpha_1\, d_t^2 x_1 + \sin\alpha_1\, d_t^2 y_1 - g\sin\alpha_1)}{\cos\alpha_1\cos(\theta - \delta\theta) - \sin\alpha_1\sin(\theta - \delta\theta)},$$

or since $y = x\tan\alpha$, $\quad y_1 = x_1\tan\alpha_1$,

$$m\cos(\alpha_1 + \theta - \delta\theta)\,.\left(\frac{d_t^2 x}{\cos\alpha} - g\sin\alpha\right)$$

$$= m'\cos(\alpha - \theta + \delta\theta)\,.\left(\frac{d_t^2 x_1}{\cos\alpha_1} - g\sin\alpha_1\right).$$

Now $$\left.\begin{aligned} x\sin(\alpha + \alpha_1) &= c\cos\alpha\sin(\alpha_1 + \theta - \delta\theta) \\ x_1\sin(\alpha + \alpha_1) &= c\cos\alpha_1\sin(\alpha - \theta + \delta\theta) \end{aligned}\right\};$$

$$\therefore\ \left.\begin{aligned} d_t^2 x\sin(\alpha + \alpha_1) &= -c\cos\alpha\cos(\alpha_1 + \theta)\, d_t^2\delta\theta \\ d_t^2 x_1\sin(\alpha + \alpha_1) &= c\cos\alpha_1\cos(\alpha - \theta)\, d_t^2\delta\theta \end{aligned}\right\},$$

omitting small quantities of a higher order than the first, since the oscillation is of small extent.

Therefore substituting these values in the above equation, we have

$$m'\cos(\alpha - \theta + \delta\theta)\,.\left\{\frac{c\cos(\alpha - \theta)}{\sin(\alpha + \alpha_1)}\, d_t^2\delta\theta - g\sin\alpha_1\right\}$$

$$+ m\cos(\alpha_1 + \theta - \delta\theta)\,.\left\{\frac{c\cos(\alpha_1 + \theta)}{\sin(\alpha + \alpha_1)}\, d_t^2\delta\theta + g\sin\alpha\right\} = 0;$$

$$\therefore \frac{mc\cos^2(\alpha_1+\theta)+m'c\cos^2(\alpha-\theta)}{\sin(\alpha+\alpha_1)}\,d_t^2\delta\theta$$

$$+g\,.\,\{m\sin\alpha\sin(\alpha_1+\theta)+m'\sin\alpha_1\sin(\alpha-\theta)\}\,\delta\theta$$

$$=g\,\{-m\sin\alpha\cos(\alpha_1+\theta)+m'\sin\alpha_1\cos(\alpha-\theta)\},$$

or $d_t^2\delta\theta$

$$+\frac{g}{c}\,\frac{\{m\sin\alpha\sin(\alpha_1+\theta)+m'\sin\alpha_1\sin(\alpha-\theta)\}\sin(\alpha+\alpha_1)}{m\cos^2(\alpha_1+\theta)+m'\cos^2(\alpha-\theta)}\,.\,\delta$$

$$=g\sin(\alpha+\alpha_1)\,.\,\frac{m'\sin\alpha_1\cos(\alpha-\theta)-m\sin\alpha\cos(\alpha_1+\theta)}{m\cos^2(\alpha_1+\theta)+m'\cos^2(\alpha-\theta)}$$

$$\text{or}\quad d_t^2\delta\theta+n^2\delta\theta=A.$$

The solution of this differential equation of the second order is

$$\delta\theta=\frac{A}{n^2}+\kappa\cos(nt+\lambda).$$

Now this expression in consequence of the circular form which it involves is periodical, and goes through all its values while nt is increased by π, or t by $\frac{\pi}{n}$, which is therefore the time of a small oscillation, and if l be the length of the simple isochronous pendulum,

$$\frac{\pi}{n}=\pi\sqrt{\frac{l}{g}},\quad\text{and}\quad l=\frac{g}{n^2};$$

$$\therefore\ l\ \frac{mc\cos^2(\alpha_1+\theta)+m'c\cos^2(\alpha-\theta)}{\{m\sin\alpha\sin(\alpha_1+\theta)+m'\sin\alpha_1\sin(\alpha-\theta)\}\sin(\alpha+\alpha_1)}.$$

Now depth of centre of gravity

$$=\frac{m\,AP\,.\sin\alpha+m'\,AP'\sin\alpha_1}{m+m'},$$

$$\text{or}\ (m+m')\,GH$$

$$=\frac{mc\sin\alpha\sin(\alpha_1+\theta)+m'c\sin\alpha_1\sin(\alpha-\theta)}{\sin(\alpha+\alpha_1)};$$

$$\therefore\ l = \frac{m c^2 \cos^2(\alpha_1 + \theta) + m' c^2 \cos^2(\alpha - \theta)}{(m + m')\, GH \,.\, \sin^2(\alpha + \alpha_1)},$$

$$\text{and}\ c^2 \cos^2(\alpha - \theta) = OP'^2 \sin^2(\alpha + \alpha_1);$$

$$\therefore\ OP'^2 = \frac{c^2 \cos^2(\alpha - \theta)}{\sin^2(\alpha + \alpha_1)},$$

$$\text{and}\ OP^2 = \frac{c^2 \cos^2(\alpha_1 + \theta)}{\sin^2(\alpha + \alpha_1)};$$

$$\therefore\ l \quad \frac{m \,.\, OP^2 + m' \,.\, OP'^2}{(m + m')\, GH}.$$

Problem XVII.

A seed grain is planted, and when one year old produces the next year ten-fold: and when two years old and upwards produces annually eighteen-fold. Every grain is planted as soon as it is produced, and the produce of each is according to the above law. Find the number of grains at the end of the x[th] *year.*

Let u_x = number at the end of the x^{th} year.

Then

$$u_{x+1} - u_x = 10\,(u_{x-1} - u_{x-2}) + 18\,u_{x-2},$$

$$\text{or}\ u_{x+1} - u_x - 10u_{x-1} - 8\,u_{x-2} = 0.$$

$$\text{Let}\ u_x = m^x.$$

Then $m^3 - m^2 - 10m - 8 = 0$,

the roots of which are 4, -1, and -2;

$$\therefore\ u_x = a \,.\, 4^x + b\,(-1)^x + c\,(-2)^x.$$

Now

$$\begin{aligned} u_0 &= a + b + c = 0 \\ u_1 &= 4a - b - 2c = 1 \\ u_2 &= 16a + b + 4c = 11 \end{aligned}$$

$$a = \frac{7}{15}, \qquad b = -\frac{9}{5}, \qquad c = \frac{4}{3};$$

$$u_x = \frac{7 \cdot 4^x - 27(-1)^x + 20(-2)^x}{15}$$

PROBLEM XVIII.

A plane is moved so as to cut off from the co-ordinate planes areas whose sum is always equal to $2n^2$: *shew that the surface, to which the plane is always a tangent plane, is represented by the equation,*

$$(3x - u) \cdot (3y - u) \cdot (3z - u) + 9n\, u = 0,$$

where $u = x + y + z \mp \sqrt{3n^2 + x^2 + y^2 + z^2 - xy - xz - yz}$.

Let $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$ be the equation to the moveable plane, a, b, c, being the parameters connected by the equation

$$ab + ac + bc = 4n^2.$$

Then differentiating we have

$$b + c + (a + b)\, d_a c = 0,$$

$$\frac{x}{a^2} + \frac{z}{c^2}\, d_a c = 0;$$

$$\therefore\ b + c = (a + b) \cdot \frac{c^2}{a^2} \cdot \frac{x}{z}.$$

Similarly $\qquad a + c = (a + b) \cdot \frac{c^2}{b^2} \cdot \frac{y}{z},$

between which and the two previous equations we must eliminate a, b, c.

Now
$$\frac{x}{a} = \frac{z}{c} \cdot \frac{b+c}{a+b} \cdot \frac{a}{c},$$

$$\frac{y}{b} = \frac{z}{c} \cdot \frac{a+c}{a+b} \cdot \frac{b}{c};$$

$$\therefore\ 1 = \frac{x}{a} + \frac{y}{b} + \frac{z}{c} = \frac{8n^2z}{c^2(a+b)};$$

$$\therefore\ \left.\begin{aligned} c^2(a+b) &= 8n^2z \\ b^2(a+c) &= 8n^2y \\ a^2(b+c) &= 8n^2x \end{aligned}\right\}\ldots\ldots\ldots\ldots\ldots\ldots(1).$$

From equations (1) we get

$$0 = y(a-c) + x(c-b) + z(b-a)\ldots\ldots\ldots\ldots(2),$$

$$2(x^2+y^2+z^2-xy-xz-yz) = x(a-c)+y(b-a)+z(c-b)\ldots(3).$$

From (1) we have

$$c^2(a+b) - a^2(b+c) = 8n^2(z-x),$$

$$\left.\begin{aligned} \text{or }\ c - a &= 2(z-x) \\ \text{So }\ b - c &= 2(y-z) \end{aligned}\right\}\ldots\ldots\ldots\ldots\ldots\ldots(4);$$

$$\frac{8n^2x}{a^2} = 2a + 4(z-x) + 2(y-z);$$

$$\frac{4n^2x}{a^2} = a + y + z - 2x;$$

$$\frac{4n^2x}{a} = a^2 + (y+z-2x)a.$$

So
$$\frac{4n^2y}{b} = b^2 + (x+z-2y)b,$$

$$\frac{4n^2z}{c} = c^2 + (x+y-2z)c;$$

$$\therefore\ 12n^2 = (a+b+c)^2$$

$$+ (y+z-2x)a + (x+z-2y)b + (x+y-2z)c$$

$$= (a+b+c)^2 + c(y-z) + b(x-y) + a(z-x)$$

$$- \{c(z-x) + b(y-z) + a(x-y)\}$$

$$= (a+b+c)^2 + c\frac{b-c}{2} + b\frac{a-b}{2} + a\frac{c-a}{2}$$

$$-2(x^2+y^2+z^2-xy-xz-yz), \text{ from (3) and (4);}$$

$$\therefore\ 2(3n^2+x^2+y^2+z^2-xy-xz-yz) = \frac{(a+b+c)^2}{2};$$

$$\frac{a+b+c}{2} = \pm\sqrt{3n^2+x^2+y^2+z^2-xy-xz-yz}.$$

Again, from (1)

$$8n^2(x+y+z) = c^2(a+b) + b^2(a+c) + a^2(b+c)$$

$$= ac(a+b+c) + bc(a+b+c) + ab(a+b+c) - 3abc$$

$$= \pm 8n^2(\sqrt{3n^2+x^2+y^2+z^2-xy-xz-yz}) - 3abc;$$

$$\therefore\ 8n^2u = -3abc \ldots\ldots\ldots\ldots\ldots\ldots (5).$$

From (1)

$$512n^6xyz = a^2b^2c^2\{8n^2(x+y+z-u) - abc\}$$

$$= \frac{64n^4u^2}{9}\left\{8n^2(x+y+z-u) + \frac{8n^2u}{3}\right\}, \text{ from (5);}$$

$$\therefore\ 27xyz = u^2\{3(x+y+z) - 2u\},$$

the equation to the surface.

Now $u^3 - 3u^2(x+y+z) + 6u^2(x+y+z) - 3u^3 = 27xyz$,

or $u^3 - 3u^2(x+y+z)$

$$+ 3u\{3n^2 + 2u(x+y+z) - u^2\} - 27xyz = 9n^2u.$$

Now $3(xy+xz+yz) = 3u\{3n^2 + 2u(x+y+z) - u^2\}$;

$$\therefore\ u^3 - 3u^2(x+y+z) + 9u(xy+xz+yz) - 27xyz = 9n^2u,$$

$$\text{or}\ (3x-u).(3y-u).(3z-u) + 9n^2u = 0.$$

Problem XIX.

A semi-elliptical arch stands with its plane vertical and perpendicular to the meridian, and its axis in the plane of the meridian; find the equation to the shadow of the arch on the horizontal plane at six o'clock, in terms of the declination of the sun and the latitude of the place.

Fig. 53. Let the plane of yz be that of the meridian, $OA = a$, $OB = b$ the semiaxes of the elliptical arch, E the pole of the great circle ZP, Z being the zenith and P the pole of the heavens; then since ZPS is a right angle, the hour being six o'clock, PS produced will meet the great circle ZE in E.

Let δ, l be the declination of the sun and the latitude of the place.

The problem is reduced to that of finding the section of the cylinder whose base is the ellipse AB and generating line parallel to SO, made by the plane xz. Let α, β, γ be the inclinations of this line to the axes of x, y, z. The equation to a generating line will be

$$\frac{X - x}{\cos\alpha} \quad \frac{Y - y}{\cos\beta} \quad \frac{Z - z}{\cos\gamma}$$

and when $Z = 0$,

$$Y = y - \frac{\cos\beta}{\cos\gamma} z, \qquad X = x - \frac{\cos\alpha}{\cos\gamma} z.$$

But
$$Y^2 = b^2 - \frac{b^2}{a^2} X^2;$$

$$(y\cos\gamma - z\cos\beta)^2 = b^2\cos^2\gamma - \frac{b^2}{a^2}(x\cos\gamma - z\cos\alpha)^2.$$

This is the equation to the cylinder formed by the sun's rays. The equation to the shadow will therefore be found from this by putting $y = 0$, and is therefore

$$z^2\left(\cos^2\beta + \frac{b^2}{a^2}\cos^2\alpha\right) - 2\frac{b^2}{a^2}\cos\alpha\cos\gamma\,.\,xz + \frac{b^2}{a^2}\cos^2\gamma\,.\,x^2$$
$$= b^2\cos^2\gamma.$$

Now $\cos\alpha = \cos SE = \cos\delta,$

$\cos\beta = \cos SZ = \cos ZP\,.\cos PS = \sin l\sin\delta,$

$\cos\gamma = \cos SD = \cos SP\,.\cos PD = \cos l\sin\delta.$

Hence the equation to the shadow is

$$b^2\cos^2 l\sin^2\delta\,.\,x^2 - 2b^2\cos l\sin\delta\cos\delta\,.\,xz$$
$$+ (a^2\sin^2 l\sin^2\delta + b^2\cos^2\delta)\,z^2 = a^2b^2\cos^2 l\sin^2\delta,$$

which is the equation to an ellipse whose centre is in O.

Problem XX.

Each of a series of numbers beginning from unity is formed by multiplying the preceding one by a, *and adding* 2m, *or subtracting* m *according as the order of the number to be formed is odd or even; shew that the* x^{th} *number in the series is equal to*

$$\frac{a^2 - 1 - m(a-2)}{a^2-1}\,.\,a^{x-1} - \frac{m}{2}\left\{\frac{1}{a-1} + \frac{3(-1)^x}{a+1}\right\}.$$

Let $u_x = x^{th}$ number.

Then $u_{x+1} - au_x = \frac{m}{2} + \frac{3m}{2}(-1)^x.$

This is a linear equation of the first order, and its solution is (Hymers' *Finite Differences*, Art. 74.)

$$\frac{u_x}{a^{x-1}} = C + \Sigma\left\{\frac{m}{2a^x} + 3m\,\frac{(-1)^x}{a^x}\right\}.$$

Let $\quad$ [illegible] $= a.$

Then $\quad u_x a^{x-1} = \frac{m}{2} \Sigma \left\{a^x + 3(-a)^x\right\} + C$

$$= \frac{m}{2}\left\{\frac{a^x}{a-1} - \frac{3}{a+1}(-a)^x\right\} + C,$$

and $\quad \frac{u_x}{a^{x-1}} = -\frac{m}{2}\left\{\frac{1}{a-1} + \frac{3(-1)^x}{a+1}\right\} \cdot \frac{1}{a^{x-1}} + C.$

Let $\quad x = 1,$

$$1 = -\frac{m}{2}\left\{\frac{1}{a-1} - \frac{3}{a+1}\right\} + C.$$

Hence $\quad \frac{u_x}{a^{x-1}} - 1 = -\frac{m}{2}\left\{\frac{1}{a-1} + \frac{3(-1)^x}{a+1}\right\} \frac{1}{a^{x-1}}$

$$-\frac{m}{2} \cdot \left\{\frac{3}{a+1} - \frac{1}{a-1}\right\},$$

$$u_x = \frac{a^2 - 1 - m(a-2)}{a^2 - 1} \cdot a^{x-1} - \frac{m}{2}\left\{\frac{1}{a-1} + \frac{3(-1)^x}{a+1}\right\}.$$

MR STEVENTON'S

AND

MR THURTELL'S PAPER,

JANUARY 10, 1840.

PROBLEM I.

Between a *and* b, a *being* < b, *insert* n *means*, a_1, a_2... a_m, *such that* $a_1 - a$, $a_2 - a_1$ .... $b - a_n$, *form an arithmetical progression whose common difference is* d: *and find the limits between which the value of* d *must lie.*

Taking the sum of the above series which consists of $n + 1$ terms we have

$$b - a = \left\{2\,(a_1 - a) + nd\right\} \frac{n+1}{2},$$

$$b = (n+1)\,a_1 - na + \frac{n\,(n+1)}{2}\,d;$$

$$a_1 = \frac{b}{n+1} + \frac{na}{n+1} \quad \frac{nd}{2}$$

$$\text{and} \quad a_n = na_1 - (n-1)\,a + \frac{n\,(n-1)}{2}\,d;$$

$$a_n = \frac{nb + n^2 a}{n+1} - \frac{n^2 d}{2} - (n-1)\,a + \frac{n^2 - n}{2}\,d$$

$$= \frac{nb + a}{n+1} - \frac{nd}{2},$$

and by assigning to n all values from 1 to n, we get the required means.

In the above expression d may have any negative value; but since $a_n > a_{n-1}$, we have

$$\frac{nb+a}{n+1} > \frac{(n-1)b+a}{n} + \frac{d}{2},$$

$$\text{or } \frac{d}{2} \text{ must not exceed } \frac{b-a}{n(n+1)}.$$

Problem II.

If it be required to find the n *quantities* $x_1, x_2 \ldots x_n$ *from the* n *equations*

$x_1+x_2=a_1$, $x_2+x_3=a_2$, &c.=&c. $x_{n-1}+x_n=a_{n-1}$, $x_n+x_1=a_n$,

shew that the problem will be determinate if n *be an odd number, but indeterminate or impossible if* n *be an even number.*

We have successively

$$x_1 + x_2 = a_1,$$

$$x_1 - x_3 = a_1 - a_2,$$

$$x_1 + x_4 = a_1 - a_2 + a_3,$$

$$x_1 - x_5 = a_1 - a_2 + a_3 - a_4,$$

&c.

I. If n be odd, we have at last

$$x_1 - x_n = a_1 - a_2 + a_3 - \&c. - a_{n-1},$$

$$x_n + x_1 = a_n;$$

$$\therefore\ 2x_1 = a_1 - a_2 + a_3 - \&c. + a_n,$$

$$\text{and } x_n = a_n - \frac{a_1 - a_2 + a_3 - \&c. + a_n}{2},$$

and the problem is determinate.

II. But if n be even, we have

$$x_1 + x_n = a_1 - a_2 + a_3 - \&c. + a_{n-1},$$

and $x_1 + x_n = a_n$.

Hence the problem is impossible unless

$$a_n = a_1 - a_2 + a_3 - \&c. + a_{n-1},$$

and this condition being fulfilled, the problem is indeterminate, since any values whatever of x_1 and x_n which make their sum equal to the above quantity, will satisfy the conditions of the problem.

Problem III.

A number of spheres are put into a hollow cone one above another, and each touching the one above, the one below, and the sides of the cone. The radii of the spheres form a geometrical progression of which the common ratio is tan² (45° + ¼ the vertical angle of the cone.)

Fig. 37. Let a be the semi-vertical angle of the cone of which and the spheres the figure is a section made by the plane of the paper passing through its axis, $PM = R$, $OL = r$.

Then $(R + r)\sin a = (VP - VO)\sin a = R - r$;

$$\therefore \frac{R+r}{R-r} = \frac{1}{\sin a},$$

$$\frac{r}{R} = \frac{1 - \sin a}{1 + \sin a} = \frac{2\sin^2\left(45^0 + \dfrac{a}{2}\right)}{2\cos^2\left(45^0 + \dfrac{a}{2}\right)}$$

$$= \tan^2\left(45^0 + \frac{a}{2}\right),$$

and the same ratio exists between any two successive radii.

Problem IV.

Shew how to find the distance of an inaccessible object upon a horizontal plane by means of a measuring chain, or staff, alone, and without observing any angles.

Fig. 38. Let D be the object, B the place of the spectator. Measure a distance BC in any direction. Place the staff at B, and measure off a distance BA, always keeping the foot of the object covered by the foot of the staff; do the same at C, and measure $AC = \delta$, $BE = \delta'$.

Let $AB = a$, $BC = b$, $CE = c$, $AE = d$.

Then $$\cos ABC = \frac{a^2 + b^2 - \delta^2}{2ab},$$

$$\cos BCE = \frac{b^2 + c^2 - \delta'^2}{2bc}.$$

Then $$BD = -b \cdot \frac{\sin BCE}{\sin(BCE + ABC)},$$

which gives the required distance, if we substitute for BCE, ABC the values already found.

Problem V.

A straight line of given length 2c *is made to move so that its ends are always in contact with two other straight lines which include a given angle 2α: shew that the locus of its middle point is an ellipse whose semi-axes are* c tan α *and* c cot α, *and the direction of one of its axes bisects the angle included by the given straight lines.*

Fig. 39. Let AC, AB be the two fixed lines including an angle $CAB = 2\alpha$: A the origin, AB and a perpen-

dicular to it the axes of co-ordinates: $AM = x$, $MP = y$ draw PN through the middle point of CB, parallel t(AC.

$$\text{Then} \quad c^2 = PN^2 + NB^2 - 2\,PN\,.\,NB\,.\cos 2\alpha,$$

$$\text{and } PN \sin 2\alpha = y, \qquad NB = AN = x - y\cot 2\alpha\,;$$

$$\therefore\; c^2 \sin^2 2\alpha = y^2 + (x \sin 2\alpha - y\cos 2\alpha)^2 - 2y\cos 2\alpha \times (x\sin 2\alpha - y\cos 2\alpha)$$

$$= x^2\sin^2 2\alpha + y^2(1 + 3\cos^2 2\alpha) - 4xy\sin 2\alpha\cos 2\alpha$$

$$\text{Now, let} \qquad \left.\begin{array}{l} x = x'\cos\theta - y'\sin\theta \\ y = x'\sin\theta + y'\cos\theta \end{array}\right\}.$$

Then substituting these values in the above equation it may be reduced to

$$Ay'^2 + Bx'^2 = (c\sin 2\alpha)^2,$$

where

$$A = \cos^2\theta\,(1 + 3\cos^2 2\alpha) + \sin^2\theta\sin^2 2\alpha + 4\sin\theta\cos\theta\sin 2\alpha\cos 2\alpha$$

$$B = \sin^2\theta\,(1 + 3\cos^2 2\alpha) + \cos^2\theta\sin^2 2\alpha - 4\sin\theta\cos\theta\sin 2\alpha\cos 2\alpha$$

under the condition,

$$\sin 2\theta\,.\,4\cos^2 2\alpha = 4\cos 2\theta\sin 2\alpha\cos 2\alpha,$$

$$\text{or } \tan 2\theta = \tan 2\alpha,$$

$$\text{or } \theta = \alpha \ldots\ldots\ldots\ldots\ldots\ldots(1),$$

$$\text{Also} \qquad A + B = \sin^2 2\alpha + 1 + 3\cos^2 2\alpha = 2(1 + \cos^2 2\alpha),$$

$$A - B = \cos 2\alpha\,(1 + 3\cos^2 2\alpha) - \cos 2\alpha\sin^2 2\alpha + 8\sin\alpha\cos\alpha\sin 2\alpha\cos 2\alpha$$

$$= 4\cos^3 2\alpha + 4\sin^2 2\alpha\cos 2\alpha$$

$$= 4\cos 2\alpha\,;$$

$$\therefore\; \left.\begin{array}{l} A = (1 + \cos 2\alpha)^2 = 4\cos^4\alpha \\ B = (1 - \cos 2\alpha)^2 = 4\sin^4\alpha \end{array}\right\}.$$

Hence the semi-axes are

$$\frac{c \sin 2a}{2 \cos^2 a}, \quad \text{and } \frac{c \sin 2a}{2 \sin^2 a},$$

that is, $c \tan a$, and $c \cot a$, and equation (1), shews that the direction of one of the axes bisects the angle between the lines.

Problem VI.

If (θ) be the angle between the tangent and focal distance at any point of an ellipse, the distance of that point from the centre $= \sqrt{a^2 - b^2 \cot^2 \theta}$.

Taking the usual notation we have from the triangle SPH,

$$4(a^2 - b^2) = SP^2 + HP^2 + 2SP \,.\, HP \cos 2\theta$$

$$= 4a^2 - 4CD^2 \sin^2 \theta;$$

$$\therefore b^2(1 + \cot^2 \theta) = a^2 + b^2 - (\text{dist.})^2 \text{ from centre};$$

$$\therefore \text{dist. from centre} = \sqrt{a^2 - b^2 \cot^2 \theta}.$$

Problem VII.

If $\alpha\beta$, $\alpha'\beta'$ be the co-ordinates of two points in a diameter of an ellipse, and be subject to the condition $a^2b^2 = a^2\beta\beta' + b^2\alpha\alpha'$, *where* a *and* b *are the semi-axes of the ellipse, shew that the equation to the tangents at the extremities of this diameter are*

$$a^2\sqrt{\beta\beta'} \,.\, y + b^2\sqrt{\alpha\alpha'} : x = \pm a^2b^2.$$

Let m be the tangent of the inclination of the diameter to the axis of x: so that

$$\beta = m\alpha, \quad \beta' = m\alpha'.$$

Then by the given condition

$$a^2b^2 = (a^2m^2 + b^2)\,a\,a';$$

$$\therefore\ \sqrt{m^2a^2 + b^2} = \frac{ab}{\sqrt{aa'}}.$$

Now the equations to tangents at the extremities of this diameter are (Hymers' *Conics*, Art. 137)

$$-\frac{b^2}{ma^2}x \pm \sqrt{\frac{b^4}{m^2a^2} + b^2}$$

$$-\frac{b^2}{ma^2}x \pm \frac{b}{ma}\sqrt{b^2 + ma^2},$$

or $$ma^2y + b^2x = \pm ab\sqrt{b^2 + ma^2} = \pm\frac{a^2b^2}{\sqrt{aa'}},$$

and $$\beta\beta' = m^2aa';$$

$$\therefore\ \sqrt{\frac{\beta\beta'}{aa'}}\cdot a^2y + b^2x = \pm\frac{a^2b^2}{\sqrt{aa'}};$$

$$\therefore\ a^2\sqrt{\beta\beta'}\,y + b^2\sqrt{aa'}\,x = \pm a^2b^2,$$

are the equations to the two tangents.

Problem VIII.

(α) *If* $\sin(a + \alpha x) = c\tan(b + \beta x)$ *for all values of* x, *then so long as* αx, βx *are small quantities,* x *is very nearly* $= 2\cdot\dfrac{\alpha\cot a - 2\beta\,\mathrm{cosec}\,2b}{\alpha^2 + 2\beta^2\sec^2 b}$.

(β) *Eliminate by differentiation the constants and exponential forms from the equation*

$$ae^y + be^{-y} = fe^x + ge^{-x}.$$

(γ) *Find the nature of the curve*

$$y + 1 = 2x - x^2 \pm (2 - x)^{\frac{5}{2}}$$

at the point corresponding to $x = 2$.

I. Since the equation $\sin(a + \alpha x) = c\tan(b + \beta x)$ holds good for all values of x, then making $x = 0$,

$$\sin a = c\tan b.$$

Expanding the above quantities by Maclaurin's Theorem, we have

$$\sin a + \alpha\cos a x - \alpha^2\sin a\frac{x^2}{2}$$

$$= c\left(\tan b + \beta\sec^2 b \,.\, x + 2\beta^2\sec^2 b\tan b\,.\,\frac{x^2}{2}\right),$$

omitting higher powers of αx, βx than the second. Hence

$$\frac{x}{2}(\alpha^2\sin a + 2c\beta^2\sec^2 b\tan b) = \alpha\cos a - c\beta\sec^2 b,$$

$$\frac{x}{2}\left(\alpha^2 + 2\beta^2\sec^2 b\frac{c\tan b}{\sin a}\right) = \alpha\cot a - c\beta\frac{\sec^2 b}{\sin a},$$

$$\therefore\ \frac{x}{2}(\alpha^2 + 2\beta^2\sec^2 b) = \alpha\cot a - 2\beta\operatorname{cosec} 2b,$$

since $\sin a = c\tan b$;

$$\therefore\ x = 2\,.\,\frac{\alpha\cot a - 2\beta\operatorname{cosec} 2b}{\alpha^2 + 2\beta^2\sec^2 b}.$$

II. $$ae^y + be^{-y} = fe^x + ge^{-x};$$

Let $p = d_x y$, $q = d_x^2 y$, $r = d_x^3 y$;

$$\therefore\ p(ae^y - be^{-y}) = fe^x - ge^{-x},$$

$$p^2(ae^y + be^{-y}) + q(ae^y - be^{-y}) = fe^x + ge^{-x};$$

$$\therefore\ (1 - p^2)\,.\,(ae^y + be^{-y}) = \frac{q}{p}(fe^x - ge^{-x});$$

$$\therefore\ \frac{fe^x + ge^{-x}}{fe^x - ge^{-x}} = \frac{q}{p - p^3}\text{..................}(1).$$

Again $p(1 - p^2)\,.\,(ae^y - be^{-y}) - 2pq(ae^y + be^{-y})$

$$= \frac{pr - q^2}{p^2}(fe^x - ge^{-x}) + \frac{q}{p}(fe^x + ge^{-x});$$

$$\therefore \left(\frac{q}{p}+2pq\right).(fe^{x}+ge^{-x}) = p(1-p^2).\frac{1}{p}(fe^{x}-ge^{-x})$$

$$\frac{pr-q^2}{p^2}.(fe^{x}-ge^{-x});$$

$$\frac{1-p^2+\frac{q^2-pr}{p^2}}{\frac{q}{p}+2pq} = \frac{fe^{x}+ge^{-x}}{fe^{x}-ge^{-x}} = \frac{q}{p-p^3}, \text{ from (1).}$$

III. Fig. 40. Let the origin be removed to the point P where $x=2$, $y=-1$: then the equation becomes

$$y = -x(x+2) \pm (-x)^{\frac{5}{2}};$$

$$\therefore d_x y = -2-2x \mp \frac{5}{2}(-x)^{\frac{3}{2}}.$$

$$\text{and } d_x^2 y = -2 \pm \frac{5.3}{4}(-x)^{\frac{1}{2}}.$$

Now for the values $x=0$, $y=0$ there is only one value of $d_x y$, or there is only one tangent inclined at an angle $\tan^{-1}(-2)$ to the axis of x. When x has a small negative value, there are two unequal positive values of y and two unequal negative values of $d_x^2 y$, giving two tangent branches concave to the axis of x, but positive values of x make y and $d_x^2 y$ impossible; hence there is no portion of the curve to the right of P. The point is therefore a cusp as indicated in the figure.

Problem IX.

If θ be eliminated from the equations $d_x y = \tan\theta$ *and* $d_x\theta = \dfrac{a e^{c\theta}}{\cos\theta}$ *shew that the curve represented by the resulting equation has this property, viz. the radius of curvature varies as the length of the arc.*

$$d_x y = \tan\theta, \quad d_x\theta = \frac{a e^{c\theta}}{\cos\theta}$$

$$\frac{\cos\theta}{a} e^{-c\theta}, \quad d_\theta y = \frac{\sin\theta}{a} e^{-c\theta};$$

$\therefore$ length of the arc $= \int_\theta \sqrt{(d_\theta x)^2 + (d_\theta y)^2} = -\frac{1}{ca} e^{-c\theta}$,

the point from which we measure the arc being at a distance $\frac{1}{ca}$ below the initial line.

Also $\quad d_x^2 y = a \sec^3\theta \,.\, e^{c\theta}, \quad 1 + (d_x y)^2 = \sec^2\theta$;

$$\therefore \text{ rad. of curvature} = -\frac{\{1 + (d_x y)^2\}^{\frac{3}{2}}}{d_x^2 y}$$

$$= -\frac{1}{a} e^{-c\theta} = c \times \text{length of arc.}$$

Problem X.

A straight uniform beam is placed upon two rough planes whose inclinations to the horizon are a and a', and the coefficients of friction $\tan\lambda$ and $\tan\lambda'$; shew that if θ be the limiting value of the angle of inclination of the beam to the horizon at which it will rest, W *its weight, and* R, R′ *the pressures upon the planes,*

$$2 \tan\theta = \cot(a' + \lambda') - \cot(a - \lambda)$$

and

$$\frac{R}{\cos\lambda \sin(a' + \lambda')} = \frac{R'}{\cos\lambda' \sin(a - \lambda)} = \frac{W}{\sin(a - \lambda + a' + \lambda')}.$$

The equations of equilibrium are

$$R' \cos a' + R \cos a + R \tan\lambda \sin a = W + R' \tan\lambda' \sin a \quad (1),$$

$$R' \sin a' + R' \tan\lambda' \cos a' = R \sin a - R \tan\lambda \cos a \quad (2),$$

and

$$\left.\begin{aligned}R' \cos(\alpha' + \theta) = R \cos(\alpha - \theta) + R' \tan\lambda' \sin(\alpha' + \theta)\\ + R \tan\lambda \sin(\alpha - \theta)\end{aligned}\right\} \quad (3).$$

From (1) $\dfrac{R'}{\cos\lambda'} \cos(\alpha' + \lambda') + \dfrac{R}{\cos\lambda} \cos(\alpha - \lambda) = W.$

From (2) $\dfrac{R'}{\cos\lambda'} \sin(\alpha' + \lambda') = \dfrac{R}{\cos\lambda} \sin(\alpha - \lambda).$

Hence

$$\frac{R'}{\cos\lambda'} \cos(\alpha' + \lambda') + \frac{\cos(\alpha - \lambda)}{\sin(\alpha - \lambda)} \cdot \frac{\sin(\alpha' + \lambda')}{\cos\lambda'} R' = W,$$

$$\text{or } \frac{R'}{\cos\lambda' \sin(\alpha - \lambda)} = \frac{W}{\sin(\alpha - \lambda + \alpha' + \lambda')}.$$

Also $\dfrac{R}{\cos\lambda} \cos(\alpha - \lambda) + \dfrac{\cos(\alpha' + \lambda')}{\sin(\alpha' + \lambda')} \cdot \dfrac{\sin(\alpha - \lambda)}{\cos\lambda} R = W,$

$$\text{or } \frac{R}{\cos\lambda \sin(\alpha' + \lambda')} = \frac{W}{\sin(\alpha - \lambda + \alpha' + \lambda')} = \frac{R'}{\cos\lambda' \sin(\alpha - \lambda)}.$$

Again from equation (3),

$$\frac{R}{\cos\lambda} \cos(\alpha - \theta - \lambda) = \frac{R'}{\cos\lambda'} \cos(\alpha' + \theta + \lambda');$$

$$\therefore \frac{\sin(\alpha' + \lambda')}{\sin(\alpha - \lambda)} = \frac{R}{\cos\lambda} \div \frac{R'}{\cos\lambda'} = \frac{\cos(\alpha' + \theta + \lambda')}{\cos(\alpha - \theta - \lambda)}$$

$$= \frac{\cos(\alpha' + \lambda') - \sin(\alpha' + \lambda') \tan\theta}{\cos(\alpha - \lambda) + \sin(\alpha - \lambda) \tan\theta};$$

$$\therefore 2 \tan\theta \sin(\alpha - \lambda) \sin(\alpha' + \lambda')$$

$$= \sin(\alpha - \lambda) \cos(\alpha' + \lambda') - \cos(\alpha - \lambda) \sin(\alpha' + \lambda'),$$

$$\text{or } 2 \tan\theta = \cot(\alpha' + \lambda') - \cot(\alpha - \lambda).$$

PROBLEM XI.

A cubical vessel full of fluid revolves uniformly about a vertical axis through one edge. The upper face is the lid of the vessel, and is moveable about the angular point next the axis; find the angular velocity that the weight of the lid may keep it in its place.

Let the angular point above-mentioned be the origin, the fixed axis the axis of z, and the two adjacent edges axes of x and y: ρ the density of the fluid; ω the angular velocity about the axis: x, y, z co-ordinates of any point in the mass of fluid; p the pressure at that point, a = a side of the lid.

Now the resolved parts of the centrifugal force on the particle (x, y, z) parallel to the axes are $\omega^2 x$, $\omega^2 y$: hence the pressure is given by the equation

$$dp = \rho \left\{\omega^2 (x\,dx + y\,dy) + g\,dz\right\}.$$

Poisson's *Mécanique*, Art. 583.

$$\text{Hence } p = \rho \left\{\frac{\omega^2}{2} (x^2 + y^2) + gz\right\} + C,$$

and when $x = 0$, $y = 0$, $z = 0$, we have $p = 0$;

$\therefore$ $C = 0$, and pressure on an element of the lid

$$= \frac{\rho\omega^2}{2} (x^2 + y^2)\, \delta x \delta y \text{ ultimately.}$$

Moment of this about axis of x

$$\frac{\rho\omega^2}{2} (x^2 + y^2)\, y\, \delta x \delta y,$$

and moment of whole pressure $= \dfrac{\rho\omega^2}{2} \int_x \int_y (x^2 y + y^3)$

$$= \frac{5}{24} \rho\omega^2 a^5,$$

the integral being taken between limits

$$x = 0, \qquad x = a; \qquad y$$

Now if σ be the density, t the thickness of the lid, the moment of its weight about the axis of x

$$= g\sigma \,.\, a^2 t \,.\, \frac{a}{2}.$$

Therefore if the lid be just in equilibrium

$$\frac{5}{24}\rho\omega^2 a^5 = g\sigma \,.\, a^2 t \,.\, \frac{a}{2},$$

$$\text{or } \omega^2 = \frac{gt}{a^4} \,.\, \frac{12\sigma}{5\rho}.$$

PROBLEM XII.

In any straight line QR, *which cuts the rectangular axes* AQx, ARy *in* Q *and* R, *a point* P *is taken such that* PQ ÷ PR *is always constant. Find the curve which always touches* QR *at the point* P, *explain the meaning of the arbitrary constant introduced, and shew that, for the same constants, there are two curves which answer the conditions of the problem.*

Fig. 41. Let $AM = x$, $MP = y$ be the co-ordinates of P. Then if $p = d_x y$,

$$PQ = \frac{y\sqrt{1+p^2}}{-p}$$

taking the negative sign because $\tan PQM = -p$, and

$$PR = x\sqrt{1+p^2};$$

$$\therefore \frac{PQ}{RP} = -\frac{y}{px} = \frac{1}{n} \text{ suppose};$$

$$\therefore \frac{d_x y}{y} = -\frac{n}{x},$$

$$\log_e y = \log_e C - n \log_e x;$$

$$\therefore y = \frac{C}{x^n}, \qquad \text{or } yx^n = C \ldots\ldots\ldots\ldots\ldots (1).$$

The constant is the value of the ordinate when $x = 1$.

Now if a point P' be taken so that $P'R = PQ$, then the value of the ratio $\frac{P'R}{P'Q}$ will be the same as that of $\frac{PQ}{PR}$. Let x, y be the co-ordinates of P'.

Then $$P'R = x\sqrt{1+p^2},$$

$$\text{and } P'Q = \frac{y\sqrt{1+p^2}}{-p};$$

$$\therefore \frac{-px}{y} = \frac{1}{n},$$

n having the same value as before;

$$\therefore \frac{n\,d_x y}{y} = -\frac{1}{x},$$

$$\text{or } \log_e y^n = \log_e C - \log_e x,$$

$$\text{or } y^n x = C \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2)$$

We observe that both the curves (1) and (2) are hyperbolic, and that the axes of co-ordinates are asymptotes. If the given ratio had been $\frac{RP}{RQ}$, we should have arrived at two parabolic curves of the n^{th} order, of which the fixed lines are the axes.

Problem XIII.

A weight is attached to a fixed point on a smooth horizontal plane by an elastic string, and is made to revolve on the plane about the fixed point; supposing the initial position of the string to be a straight line, and the original velocity to be impressed in a direction perpendicular to that of the string, find the path of the weight.

Let the fixed point be the origin, x, y the rectangular, r, θ the polar co-ordinates of the particle (m) at the time t from the commencement of the motion, T the tension of the string at that time. Then the equations of motion are

$$\left.\begin{aligned} m d_t^2 x &= -T.\frac{x}{r} \\ m d_t^2 y &= -T.\frac{y}{r} \end{aligned}\right\} \quad ,(1).$$

Also $x^2 + y^2 = r^2$ and $r = a(1 + \epsilon T)$, a being the original length of the string, and ϵ its extensibility*.

From (1) $\quad x d_t^2 y - y d_t^2 x = 0,$

$$\left.\begin{aligned} \text{or } x d_t y - y d_t x &= au \\ \therefore\ r^2 d_t\theta &= au \end{aligned}\right\} \ldots\ldots\ldots\ldots\ldots\ldots(2),$$

u being the original velocity impressed on m.

Again, from (1)

$$2m d_t^2 x d_t x + 2m d_t^2 y d_t y = -2T d_t r,$$

$$\text{or } m d_t\{(d_t x)^2 + (d_t y)^2\} = -2 d_t r.\frac{r-a}{a\epsilon}$$

$$\therefore\ m d_\theta\{(d_\theta r)^2 + r^2\}.\frac{(au)^2}{r^4} = -2 d_\theta r.\frac{r-a}{a\epsilon}, \text{ from (2).}$$

* The form of the string will evidently not deviate from a straight line, as there are no forces acting on it transversely.

Now let $r = \frac{1}{v}$. Then $d_\theta r = -\frac{d_\theta v}{v^2}$;

$$\therefore\ m(d_\theta^2 v + v)\cdot(au)^2 = \frac{1}{a\epsilon}\left(\frac{1}{v^3} - \frac{a}{v^2}\right),$$

$$m(au)^2\left\{(d_\theta v)^2 + v^2\right\} = \frac{2}{a\epsilon}\left(\frac{a}{v} - \frac{1}{2v^2}\right) + C,$$

and when $v = \frac{1}{a}$, $\qquad d_\theta v = 0$;

$$\therefore\ m(au)^2\cdot\frac{1}{a^2} = \frac{2}{a\epsilon}(a^2 - \tfrac{1}{2}a^2) + C;$$

$$\therefore\ m(au)^2\left\{(d_\theta v)^2 + v^2\right\} = \frac{2}{a\epsilon}\left(\frac{a}{v} - \frac{1}{2v^4}\right) + mu^2 - \frac{a}{\epsilon},$$

ϵ being of -4 linear dimensions; and this is the simplest form of the equation to the path of m. If $v^2 = x$, the above equation is of the form

$$d_\theta x = 2\sqrt{A\sqrt{x} + Bx - x^2 + C},$$

which cannot be integrated.

Problem XIV.

The moment of inertia of any plane figure about any axis equally inclined to the principal axes which have the same origin, is equal to two thirds of the greatest of the moments about those principal axes.

Let A, B, C be the moments of inertia about three principal axes, C being about an axis perpendicular to the plane figure.

Then moment of inertia (Q) about an axis equally inclined to them $= \frac{1}{3}(A + B + C)$ (Earnshaw's *Dynamics*. Art. 193). Now if x, y be the co-ordinates of an element δm of the plane, then

$$\begin{aligned} C &= \Sigma\delta m\,(x^2 + y^2) \\ &= \Sigma\delta m \,.\, x^2 + \Sigma\delta m \,.\, y^2 \\ &= A + B \end{aligned}$$

the axes of x and y being principal axes;

$$\begin{aligned} \therefore\ Q &= \tfrac{2}{3}C \\ &= \tfrac{2}{3}\text{ greatest moment.} \end{aligned}$$

since $C > A > B$.

Problem XV.

A uniform and straight plank rests with its middle point upon a rough horizontal cylinder, their directions being perpendicular to each other. Find the greatest weight that can be put upon one end of the plank without its sliding off the cylinder. Also, supposing the weight suddenly removed, find the time of the small oscillations of the plank.

Fig. 42. The figure represents a section of the plank and cylinder made by the plane of the paper, which is perpendicular to the axis of the cylinder. Let R and $\tan\lambda . R$ be the reaction and friction at P: w, W the weights of the plank and the attached weight, which is such that the plank is on the point of sliding: θ the inclination of the plank to the horizon, or of PO to the vertical: $2a$ the length of the plank, and r the radius of the cylinder.

Then $R\sin\theta = R\tan\lambda\cos\theta$..............(1),

$$w \,.\, GP = W \,.\, PB \text{.....................(2)},$$

$$R\tan\lambda\sin\theta + R\cos\theta = W + w \text{......(3)}.$$

From (1) $\qquad \tan\theta = \tan\lambda.$

Also $\qquad GP = r\lambda;$

$$\therefore\ w \cdot r\lambda = W(a - r\lambda) \text{ from (2)};$$

$$\therefore\ \frac{W}{\omega} = \frac{r\lambda}{a - r\lambda} \text{ which gives } W.$$

II. Again, supposing the weight suddenly removed, and θ to be the inclination of the plank to the horizon at the time t, we have the following equations of motion, in which x, y are the horizontal and vertical co-ordinates of G, reckoned from O.

$$m d_t^2 x = R \sin\theta - \mu R \cos\theta,$$

$$m d_t^2 y = R \cos\theta + \mu R \sin\theta - mg,$$

$$m k^2 d_t^2 \theta = - R r \theta,$$

$$\left.\begin{array}{l} x = r \sin\theta - r\theta \cos\theta \\ y = r \cos\theta + r\theta \sin\theta \end{array}\right\} \dots\dots\dots\dots\dots (1);$$

$$\therefore\ m(d_t^2 x \sin\theta + d_t^2 y \cos\theta) = R - mg \cos\theta.$$

Now differentiating the equations (1) we find that $d_t^2 x$ is a quantity of the third order supposing the oscillation to be of small extent; and $d_t^2 y$ is of the second order: hence approximately $R = mg$;

$$\therefore\ d_t^2\theta + \frac{gr}{k^2}\theta = 0 \dots\dots\dots\dots\dots\dots (2),$$

$$\text{and time of a small oscillation} = \frac{\pi k}{\sqrt{gr}},$$

where k is the radius of gyration of the plank about an axis through its centre of gravity and parallel to the axis of the cylinder.

Obs. We might have obtained equation (2) at once by taking the moments about P considered as an instantaneous axis.

Problem XVI.

A telegraph has m *arms, and each arm is capable of* n *distinct positions: find the total number of signals which can be made with the telegraph.*

One arm is capable of n positions, and there are m arms;

$\therefore\ mn =$ number of signals by using only 1 arm,

$n^2 =$ .... any *particular* two;

$\therefore\ \dfrac{m(m-1)}{2} n^2 =$ number of signals by using 2 arms at once.

Also

$n^3 =$ number of signals by using any *particular* three;

$$\therefore\ m\frac{m-1}{2}\cdot\frac{m-2}{3}n^3$$

$=$ number of signals by using any 3 arms at once,

&c...............................

and $n^m =$ number of signals by using m arms at once; therefore the whole number of signals

$$= \left\{1 + mn + m\frac{m-1}{2}n^2 + m\frac{m-1}{2}\cdot\frac{m-2}{3}n^3 + \ldots + n^m\right\} - 1$$

$$= (1+n)^m - 1.$$

Problem XVII.

The whole height of the fall at a water-mill, and the quantity of water passing through the wheel in a given time, are given; find the relation between the height from which the water falls before it strikes the buckets, and the uniform velocity of the centre of each

bucket, when the effect produced is the greatest possible. The breadth of the buckets or the thickness of the vein of water which acts upon the wheel is supposed small in comparison with the radius of the wheel: and the effect of backwater is neglected.

Let H = whole height of the water-fall,

h = height through which the water falls before it reaches the bucket.

v = uniform velocity of the circumference of the wheel.

The velocity lost by the impulse $= \sqrt{2gh} - v$, and its effect is measured by $m(\sqrt{2gh} - v)v$. (Navier, *sur les Machines, Troisième Partie*, No. 112.) After the impulse and during the remainder of its actions the effect is measured by $mg(H - h)$, m being the mass of water discharged in an unit of time.

The whole effect is therefore measured by

$$mg(H - h) + m(\sqrt{2gh} - v)v,$$

and this is to be a maximum by the variation of v: when this is the case

$$\sqrt{2gh} = 2v,$$

$$\text{and } v^2 = \tfrac{1}{2}gh,$$

and this is the relation required.

If v is given the relation is $v^2 = 2gh$.

Problem XVIII.

A sphere of less specific gravity than water is placed at a given depth in a stream running with a given uniform velocity, and then left to the action of the stream; find the motion and path of the sphere.

Let u = the horizontal velocity of the stream, x, y the horizontal and vertical co-ordinates of the centre of the sphere at time t, R the resistance, considered as a retarding force,

$$= K\left\{(u - d_t x)^2 + (d_t y)^2\right\},$$

g the weight of the sphere in water.

The equations of motion are

$$d_t^2 x = R\,d_s x, \quad d_t^2 y = R\,d_s y + g.$$

Then $d_x^2 y = \dfrac{1}{(d_t x)^3}\left\{d_t x\, d_t^2 y - d_t y\, d_t^2 x\right\} = \dfrac{g\,d_t x}{(d_t x)^3}$;

$$\therefore\ (d_t x)^2 = \frac{g}{d_x^2 y}\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots(1)^*.$$

Also $$\frac{d_s^2 t}{(d_s t)^3} = -\frac{R}{d_s s}\ldots\ldots\ldots(2).$$

Differentiating (1), we have

$$\frac{2 d_s^2 t}{(d_s t)^3} = \frac{g\,d_x^3 y}{(d_x^2 y)^2};$$

$$\therefore\ \frac{g\,d_x^3 y}{(d_x^2 y)^2} = -\frac{2R}{d_x s} = -\frac{2K}{d_x s}\left\{u^2 - 2u\,d_t x + (d_t s)^2\right\}$$

$$\cdot\frac{2K}{d_x s}\left\{u^2 - 2u\sqrt{\frac{g}{d_x^2 y}} + (d_x s)^2\cdot\frac{g}{d_x^2 y}\right\};$$

$$\therefore\ g\,d_x^3 y\sqrt{1 + (d_x y)^2}$$

$$+ 2K\,d_x^2 y\left\{u^2 d_x^2 y - 2u\sqrt{g.d_x^2 y} + g(d_x s)^2\right\} = 0.$$

This is the differential equation to the path of the centre of the sphere. It admits of one integration by

* This equation may be obtained at once from the consideration that the only forces which act in the normal are the centrifugal force and the resolved part of gravity.

putting $d_x^2 y = v^2$, involving circular and logarithmic forms; but as it leads to no result of importance, we shall not give it.

Also $$(d_t s)^2 = (d_x s)^2 . (d_t x)^2 = \frac{g(d_x s)^2}{d_x^2 y},$$

which when the path is known determines the velocity.

Now if q be the chord of curvature at (xy) parallel to the vertical,

$$q = 2 . \frac{1 + (d_x y)^2}{d_x^2 y} = \frac{2(d_x s)^2}{d_x^2 y};$$

$$\therefore \text{ (vel.)}^2 = g . \frac{q}{2},$$

or the velocity is that due to one-fourth the chord of curvature drawn parallel to the vertical.

MR THURTELL'S PAPER,

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PROBLEM I.

(α) *If the price in shillings of a cwt. of goods be multiplied by 3 and divided by 7, the result is the value in farthings of a pound weight.*

(β) *If* a_1, $a_2 \ldots a_n$ *be positive quantities,*

$$\frac{n-1}{2}(a_1 + a_2 + \ldots + a_n)$$

is always greater than

$$\sqrt{a_1 a_2} + \sqrt{a_1 a_3} + \sqrt{a_2 a_3} + \ldots$$

Also the former is the greatest value the latter expression can have under the condition $a_1 + a_2 + \ldots + a_n =$ *constant.*

(α) Let x = price of a cwt. in shillings;

$$\frac{x}{112} = \ldots\ldots\ldots\ldots \text{ lb.}$$

$$\therefore \frac{48x}{112} = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{ farthings.}$$

Now $\dfrac{48x}{112} = \dfrac{3}{7}x$, which is found from x by multiplying by 3 and dividing by 7.

(β) $(\sqrt{a_1} - \sqrt{a_2})^2$ = positive quantity;

$$\therefore a_1 + a_2 > 2\sqrt{a_1 a_2}.$$

Similarly $$a_1 + a_3 > 2\sqrt{a_1 a_3},$$
$$a_2 + a_3 > 2\sqrt{a_2 a_3},$$
$$\&c. > \&c.;$$

$$\therefore\ (n-1).(a_1+a_2+\ldots+a_n) > 2(\sqrt{a_1 a_2}+\sqrt{a_1 a_3}+\sqrt{a_2 a_3}+\ldots),$$

or $$\frac{n-1}{2}.(a_1+a_2+\ldots+a_n) > \sqrt{a_1 a_2}+\sqrt{a_1 a_3}+\sqrt{a_2 a_3}+\ldots.$$

Again, let
$$\begin{aligned} 2u = {} & \sqrt{a_1}\,(\sqrt{a_2}+\sqrt{a_3}+\ldots+\sqrt{a_n}) \\ & + \sqrt{a_2}\,(\sqrt{a_1}+\sqrt{a_3}+\ldots+\sqrt{a_n}) \\ & + \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ & + \sqrt{a_n}\,(\sqrt{a_1}+\sqrt{a_2}+\ldots+\sqrt{a_{n-1}}), \end{aligned}$$
where $a_1 + a_2 + \ldots + a_n =$ constant.

Hence a_n is a function of the $n-1$ independent variables a_1, $a_2 \ldots a_{n-1}$; differentiating and observing that $-1 = d_{a_1} a_n = d_{a_2} a_n = \&c.$, we have

$$(\sqrt{a_n}-\sqrt{a_1}).(\sqrt{a_2}+\sqrt{a_3}+\ldots+\sqrt{a_{n-1}}) = a_n - a_1,$$
$$(\sqrt{a_n}-\sqrt{a_2}).(\sqrt{a_1}+\sqrt{a_3}+\ldots+\sqrt{a_{n-1}}) = a_n - a_2,$$
$$\&c. = \&c.$$

$$\therefore\ a_n = a_1 = a_2 = \&c. = a_{n-1};$$

therefore the maximum value of u

$$\frac{n-1}{2}.(a_1+a_2+\ldots+a_n).$$

Problem II.

If there be n *equations between* n *unknown quantities* x_1, $x_2 \ldots x_n$ *and* $n+1$ *known quantities*, a_1, $a_2 \ldots a_n$ *and* c, *the* m^{th} *equation being*

$$a_1^{m-1}x_1 + a_2^{m-1}x_2 + \ldots\ldots + a_n^{m-1}x_n = c^{m-1}:$$

$$\textit{then } x_1 = \frac{(c-a_2).(c-a_3)\ldots(c-a_n)}{(a_1-a_2).(a_1-a_3)\ldots(a_1-a_n)}$$

and similarly for x_2, $x_3 \ldots$

The n equations are

$$x_1 + x_2 + x_3 + \ldots\ldots + x_n = 1 \ldots\ldots\ldots\ldots\ldots\ldots(1),$$

$$a_1x_2 + a_2x_2 + a_3x_3 + \ldots\ldots + a_nx_n = c \ldots\ldots\ldots\ldots(2),$$

$$a_1^2x_1 + a_2^2x_2 + a_3^2x_3 + \ldots\ldots + a_n^2x_n = c^2 \ldots\ldots\ldots(3),$$

$$\&c. \quad = \quad \&c.$$

$$a_1^{n-1}x_1 + a_2^{n-1}x_2 + a_3^{n-1}x_3 + \ldots\ldots + a_n^{n-1}x_n = c^{n-1}\ldots(n).$$

Multiply (1) by a_n and subtract it from (2). Then we have

$$c - a_n = (a_1 - a_n)x_1 + (a_2 - a_n)x_2 + \ldots\ldots + (a_{n-1} - a_n)x_{n-1},$$

$$\text{or } 1 = p_1x_1 + p_2x_2 + \ldots\ldots + p_{n-1}x_{n-1}\ldots\ldots\ldots\ (\alpha),$$

$$\text{where } p_1 = \frac{a_1 - a_n}{c - a_n},\quad p_2 = \frac{a_2 - a_n}{c - a_n}\ \ldots\ldots\ p_{n-1} = \frac{a_{n-1} - a_n}{c - a_n}.$$

So from the succeeding equations we get

$$c = p_1a_1x_1 + p_2a_2x_2 + \ldots\ldots + p_{n-1}a_{n-1}x_{n-1}\ldots\ldots(\beta),$$

$$c^2 = p_1a_1^2x_1 + p_2a_2^2x_2 + \ldots\ldots + p_{n-1}a_{n-1}^2x_{n-1}\ \ldots\ (\gamma),$$

$$\&c. \quad = \quad \&c.$$

$$c^{n-2} = p_1a_1^{n-2}x_1 + p_2a_2^{n-2}x_2 + \ldots\ldots + p_{n-1}a_{n-1}^{n-2}x_{n-1}\ldots(\nu),$$

which $n-1$ equations do not involve x_n.

Proceeding similarly with these we have

$$c - a_{n-1} = p_1 (a_1 - a_{n-1})\, x_1 + p_2 (a_2 - a_{n-1})\, x_2 + \ldots\ldots$$
$$+ p_{n-2} (a_{n-2} - a_{n-1})\, x_{n-2},$$

$$\text{or } 1 = p_1 q_1 x_1 + p_2 q_2 x_2 + \ldots\ldots + p_{n-2} q_{n-2} x_{n-2},$$

the $(n-2)^{\text{th}}$ such equation being

$$c^{n-3} = p_1 q_1 a_1^{\,n-3} x_1 + \ldots\ldots + p_{n-2} q_{n-2} a_{n-2}^{n-3} x_{n-2},$$

$$\text{where } q_1 = \frac{a_1 - a_{n-1}}{c - a_{n-1}}, \quad q_2 = \frac{a_2 - a_{n-1}}{c - a_{n-1}}, \ \&\text{c.} = \&\text{c.}$$

in which $n-2$ equations, neither x_n nor x_{n-1} are involved.

Hence, after $n-1$ such operations we have

$$1 = p_1 q_1 r_1 \ldots\ldots x_1,$$

$$\phantom{c - a_n \quad c - a_{n-1}} \qquad \frac{a_1 - a_2}{c - a_2}\, x_1$$
$$c - a_n \quad c - a_{n-1}$$

$$\therefore\ x_1 = \frac{(c - a_2)(c - a_3) \ldots\ldots (c - a_n)}{(a_1 - a_2)(a_1 - a_3) \ldots\ldots (a_1 - a_n)}$$

Similarly with x_2, x, &c.

Problem III.

A cloud, or other object known to be moving in a horizontal plane, is observed by a person within a room: from the line in which it seems to move across the window and the position of the eye, determine the direction of its motion.

Let the vertical plane of the window be the plane of xz: the eye being in the axis of y at a distance β from the origin. Then the intersection of the plane which passes through the eye and the line of motion of the object, with the plane of xz, will be the line in which the object appears to move across the window.

Let $\dfrac{x}{\alpha} + \dfrac{z}{\gamma} = 1$ be its equation, α, γ being known: and let $z = c$ be the equation to the horizontal plane in which the object moves.

Then the equation to the above-mentioned plane is

$$\frac{x}{\alpha} + \frac{y}{\beta} + \frac{z}{\gamma} = 1:$$

and therefore $z = c$,

$$\frac{x}{\alpha} + \frac{y}{\beta} = 1 - \frac{c}{\gamma}$$

are the equations to the line of motion; its inclination to the plane of the window being $\tan^{-1}\left(\dfrac{\beta}{\alpha}\right)$.

PROBLEM IV.

There are n *straight lines in a plane;* α *of them are parallel to each other and to none else,* β *parallel to each other and to none else, and so on; and none of the remaining* $n - \alpha - \beta - \gamma - \ldots$ *lines are parallel to any of the former, or to each other. Find the total number of their intersections.*

The number of intersections of sets (α), $(\beta) = \alpha\beta$,

the number of intersections of the set (γ) with the sets

(α), $(\beta) = \gamma(\alpha + \beta)$,

.... (δ) ....

(α), (β), $(\gamma) = \delta(\alpha + \beta + \gamma)$,

.... (ϵ) ....

(α), (β), (γ), $(\delta) = \epsilon(\alpha + \beta + \gamma + \delta)$,

&c. = &c.;

therefore whole number of intersections (N)

$$= \beta . \alpha + \gamma(\alpha + \beta) + \delta(\alpha + \beta + \gamma) + \epsilon(\alpha + \beta + \gamma + \delta) + \&c.,$$

$$\begin{aligned} \text{and } 2N = {} & \alpha(\beta + \gamma + \delta + \epsilon + \ldots) \\ & + \beta(\alpha + \gamma + \delta + \epsilon + \ldots) \\ & + \gamma(\alpha + \beta + \delta + \epsilon + \ldots) \\ & + \delta(\alpha + \beta + \gamma + \epsilon + \ldots) \\ & + \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \end{aligned}$$

$$\text{or } N = \frac{\alpha}{2}(n - \alpha) + \frac{\beta}{2}(n - \beta) + \frac{\gamma}{2}(n - \gamma) + \frac{\delta}{2}(n - \delta) + \&c.$$

Problem V.

If A *and* B *be the extremities of the axis major of a conic section,* T *the point where a tangent at a point* P *in the curve meets* AB, QTR *a line perpendicular to* AB *and meeting* AP *and* BP *in* Q *and* R *respectively; then* QT = RT.

A being the origin, let x, y be the co-ordinates of P and $y^2 = 2px + nx^2$ the equation to the conic section, n being finite. Then $AB = -\frac{2p}{n}$.

The equation to the tangent is

$$(y' - y)y = (x' - x) . (p + nx);$$

$$\therefore AT - x = \frac{-y^2}{p + nx}.$$

The equations to AP, BP are respectively

$$y' = \frac{y}{x}x' \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1),$$

$$\text{and } y' - y = \frac{ny}{2p + nx} . (x' - x) \ldots\ldots\ldots\ldots (2).$$

In these equations make $x' = AT = \dfrac{-px}{p+nx}$.

Then $$TQ = \frac{-py}{p+nx},$$

and $$TR = -y' = -y + \frac{ny}{2p+nx} \cdot \frac{y^2}{p+nx}$$

$$= -y + \frac{nxy}{p+nx}$$

$$= \frac{-py}{p+nx};$$

$$\therefore\ TR = TQ.$$

PROBLEM VI.

A and C (*fig.* 43) *are fixed points, about which beams* AB, CD *are freely moveable by hinge joints;* AB *is supported in a horizontal position by* CD, *and has a weight suspended at* B. *Find the pressure at* C, (1) *when there is a hinge joint at* D, (2) *when* CD *forms one piece with* AB, *the weights of the beams being neglected.*

Let R_x, R_y, r_x, r_y be the horizontal and vertical reactions at A and C respectively. Then whether there is a joint at D or not, we may consider $CDAB$ as a rigid body;

$$\therefore\ R_x = r_x \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1),$$

$$R_y + P = r_y \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2).$$

But I. if there is a hinge at D, in taking the moments about that point, we may consider the equilibrium of CD and AB separately, in which case

$$\left.\begin{aligned} R_y \cdot AD &= P \cdot DB \\ \text{and } r_y \cdot AD &= r_x \cdot AC \end{aligned}\right\} \ldots\ldots\ldots\ldots\ldots\ldots(3).$$

Hence we have $r_y = P \cdot \frac{DB}{AD} + P = P \cdot \frac{AB}{AD}$,

$$r_x = r_y \cdot \frac{AD}{AC} = P \cdot \frac{AB}{AC};$$

$\therefore$ pressure at $C = \sqrt{r_x^2 + r_y^2} = P \cdot AB \sqrt{\frac{1}{AD^2} + \frac{1}{AC^2}}$.

But II. if CD form one piece with AB, there is only one equation of moments, which is

$$(r_y - R_y) \cdot AD + P \cdot DB = r_x \cdot AC,$$

$$\text{or } P \cdot AB = r_x \cdot AC,$$

$$\text{and } r_x = P \cdot \frac{AB}{AC},$$

and the value of r_y is indeterminate.

Problem VII.

Two billiard balls are lying in contact on the table: in what direction must one of them be struck by a third so as to go off in a given direction? The balls are supposed equal in all respects, smooth, and of given elasticity.

Fig. 44. Let A, B be the centres of the balls in contact, C the centre of the impinging ball, θ the angle which CB produced makes with AB, BE the direction in which the ball moves off, $EBX = \alpha$.

The problem divides itself into two parts, first the motion during compression, and then the motion during restitution.

Let R, Re be the impulsive actions in these two cases between B and C, R_1 R_1e between A and B: and

in the first part of the motion, let v_1 be the initial velocity of B in a direction making an angle ϕ with AB; v_2 the velocity of A; V, v the velocities of C before and after the impact, m the mass of either ball.

Then the equations of motions are

$$m(V-v) = R, \quad (1) \qquad mv_1 \cos\phi = R\cos\theta + R_1. \quad (2)$$

$$mv_1 \sin\phi = R\sin\theta, \quad (3) \qquad mv_2 = R_1, \quad (4)$$

$$v_1 \cos(\theta - \phi) = v, \quad (5) \qquad v_1 \cos\phi = -v_2. \quad (6).$$

Then from (2), (3), (5),

$$\cos\theta\,(R\cos\theta + R_1) + R\sin^2\theta = mv = mV - R;$$

$$\therefore\ 2R + R_1\cos\theta = mV.$$

Also $\qquad R_1 = -mv_1\cos\phi = -R\cos\theta - R_1;$

$$\therefore\ 2R - \frac{R}{2}\cos^2\theta = mV;$$

$$\therefore\ R = \frac{2mV}{4-\cos^2\theta}, \text{ and } R_1 = -\frac{mV\cos\theta}{4-\cos^2\theta}.$$

In the second part of the motion, let u_1 be the value of v_1, then

$$\left.\begin{array}{l} mu_1\cos\alpha - mv_1\cos\phi = Re\cos\theta + eR_1 \\ mu_1\sin\alpha - mv_1\sin\phi = Re\sin\theta \end{array}\right\}.$$

Adding these to the former equations, we have

$$mu_1\cos\alpha = R(1+e)\cos\theta + (1+e)R_1,$$

$$mu_1\sin\alpha = R(1+e)\sin\theta;$$

$$\therefore\ \tan\alpha = \frac{R\sin\theta}{R\cos\theta + R_1} = \frac{2mV\sin\theta}{2mV\cos\theta - mV\cos\theta}$$

$$= 2\tan\theta,$$

and $\theta = \tan^{-1}\left(\dfrac{\tan\alpha}{2}\right)$ is the angle required.

Problem VIII.

If for any curve, r *be the distance of any point in it from a fixed point, and* s *the arc, then the perpendicular from the fixed point upon the normal* $= \mathrm{r d_s r}$.

If ρ *be the radius of curvature, then the radius of curvature* ρ_1 *of the evolute at the corresponding point* $= \rho \mathrm{d_s} \rho$: *and if* ρ_n *be the radius of curvature of the* $\mathrm{n^{th}}$ *evolute (the evolute of the first evolute being called the second evolute, and so on) then* $\rho_n = \rho \mathrm{d_s} \rho_{n-1}$. *Prove these formulæ and give their geometrical interpretation.*

Fig. 45. Let SY_1 be the perpendicular on the normal:

$$SP_1 = r_1, \quad SY_1 = p_1, \quad SP = r, \quad A_1P_1 = s_1, \quad PSK = \theta.$$

Then

$$Sy_1 = SP \cos PSY_1 = r \cos SPT = \frac{r d_\theta r}{\sqrt{r^2 + (d_\theta r)^2}} = r d_s r.$$

The equations connecting the involute and evolute are

$$p_1 = r d_s r \ldots\ldots\ldots\ldots\ldots\ldots (1),$$

$$\left.\begin{array}{ll} r_1^2 = r^2 + \rho^2 - 2p\rho \ (2) & \text{from triangle } SPP_1 \\ p_1^2 = r^2 - p^2 \ (3) \ldots\ldots\ldots\ldots\ldots & \quad . \quad SPY_1 \end{array}\right\}.$$

Hence, $\quad p_1 d_p p_1 = \rho - p$ from (3),

and $r_1 d_p r_1 = (\rho - p)\, d_p \rho \ldots \cdot (2)$;

$$\therefore \ \rho_1 = r_1 d_{p_1} r_1 = p_1 d_p \rho = r d_s r \, d_p \rho \ \text{from (1)}$$

$$= \frac{r d_p r}{d_p s} d_p \rho = \rho d_s \rho.$$

Again, $\quad \rho_2 = \rho_1 d_{s_1} \rho_1 \quad \dfrac{\rho d_s \rho_1}{d_s s_1} d_s \rho = \rho d_s \rho_1,$

since $\rho = s_1 + c$.

Suppose this true for ρ_n. Then

$$\rho_n = \rho d_s \rho_{n-1},$$

$$\rho_{n+1} = \rho_n d_{s_n} \rho_n = \rho\, d_s \rho_{n-1} \frac{d_s \rho_n}{d_s s_n},$$

$$\text{and } s_n = \rho_{n-1} + c;$$

$$\therefore \rho_{n+1} = \rho d_s \rho_n.$$

Hence, if it is true for ρ_n, it is true for ρ_{n+1}: but it is true for $\rho_1 \rho_2$; therefore it is generally true, and

$$\rho_n = \rho d_s \rho_{n-1}.$$

Since
$$\frac{\delta \rho_{n-1}}{\rho_n} = \frac{\delta s}{\rho},$$

$$\frac{\delta s_n}{\rho_n} = \frac{\delta s}{\rho},$$

the general formula shews that the angle subtended at the centre of curvature by any small arc of the original curve is ultimately equal to that subtended by the corresponding arc of each of the series of evolutes at its centre of curvature.

Problem IX.

Shew that a uniform rod of different density from that of water, when left to itself in water, must practically *float at length on the surface or lie at length at the bottom.*

A uniform rod, of less specific gravity than that of water, has a string attached to one end of it, and also to the bottom of a pond whose depth is less than the sum of the lengths of the rod and string; find the position of equilibrium.

If the specific gravity of the rod be greater than that of water, and the rod be placed in the fluid vertically, it will rest in unstable equilibrium, and therefore practically will not remain in that position. If the rod be placed in the fluid obliquely, the centre of gravity will descend, and will continue to descend after one end has reached the bottom, since the moment of the weight of the rod about that end is always greater than that of the fluid displaced by it. Hence it will lie in stable equilibrium on the bottom.

If the specific gravity of the rod be less than that of water, some portion of it must lie without the fluid: and with this condition it cannot lie obliquely, since the centres of gravity of the rod and fluid displaced by it could not be in the same vertical: hence, since the centre of gravity will not ascend, the rod will rest in stable equilibrium, horizontally on the surface.

Fig. 46. Let AB be the rod inclined at an angle θ to the horizon.

$AG = a \; AD = s$, $CD = d$, κ area of a section of the rod: ρ, σ the specific gravities of the fluid and rod.

Then since the string must be vertical, DAC is a straight line, and perpendicular to the horizon: therefore taking moments about A, we have

$$g\rho\frac{(AK)^2}{2}\cdot\kappa\cos\theta = g\sigma\cdot 2a^2\cdot\kappa\cos\theta,$$

$$\text{or } \rho\frac{(d-s)^2}{\sin^2\theta} = 4\sigma a^2,$$

$$\text{and } \theta = \sin^{-1}\left(\frac{d-s}{2a}\cdot\sqrt{\frac{\rho}{\sigma}}\right).$$

Problem X.

A plane revolves uniformly about a vertical axis, and a body descends with its flat smooth surface pressing against the plane: find the equation to the path of the body upon the plane.

Let x, y, z be the co-ordinates of the centre of gravity of the body m at the time t from the commencement of the motion, when the revolving plane coincided with the plane of xz; ω the angular velocity of the plane, R the reaction of the plane, r, z the co-ordinates of the body in the plane.

Then the equations of motion are

$$m d_t^2 x = -R \sin \omega t, \qquad m d_t^2 y = R \cos \omega t,$$

$$m d_t^2 z = -mg,$$

$$\text{and} \quad x = r \cos \omega t, \qquad y = r \sin \omega t;$$

$$\therefore \left.\begin{aligned} d_t^2 x &= d_t^2 r \cos \omega t - 2\omega d_t r \sin \omega t - \omega^2 r \cos \omega t \\ d_t^2 y &= d_t^2 r \sin \omega t + 2\omega d_t r \cos \omega t - \omega^2 r \sin \omega t \end{aligned}\right\} (2).$$

Now from the first two equations

$$d_t^2 x \cos \omega t + d_t^2 y \sin \omega t = 0,$$

$$\text{or} \quad d_t^2 r = \omega^2 r \text{ from } (2);$$

$$\therefore 2 d_t r \, d_t^2 r = 2\omega^2 r \, d_t r,$$

$$\text{or} \quad (d_t r)^2 = C + \omega^2 r^2,$$

$$\text{and} \quad 0 = C + \omega^2 a^2,$$

if a be the original value of r, and the body be supposed, for the sake of simplicity, originally at rest. Hence

$$\left.\begin{aligned} (d_t r)^2 &= \omega^2 (r^2 - a^2) \\ \text{and} \quad (d_t z)^2 &= 2g (b - z) \end{aligned}\right\};$$

$$\therefore (d_r z)^2 = \frac{2g}{\omega^2} \cdot \frac{b - z}{r^2 - a^2};$$

$$\therefore \frac{-d_r x}{\sqrt{b-x}} = \sqrt{\frac{2g}{\omega^2}}\, \frac{1}{\sqrt{r^2-a^2}};$$

$$\therefore 2\sqrt{b-x} = \sqrt{\frac{2g}{\omega^2}} \log_\epsilon \left\{\frac{r}{a} + \sqrt{\left(\frac{r}{a}\right)^2 - 1}\right\};$$

$$\left.\begin{aligned} \therefore\ \epsilon^{\omega\sqrt{\frac{2b-2x}{g}}} &= \frac{r}{a} + \sqrt{\left(\frac{r}{a}\right)^2 - 1} \\ \text{and } \epsilon^{-\omega\sqrt{\frac{2b-2x}{g}}} &= \frac{r}{a} - \sqrt{\left(\frac{r}{a}\right)^2 - 1} \end{aligned}\right\};$$

$$\therefore r = \frac{a}{2}\left(\epsilon^{\omega\sqrt{\frac{2b-2x}{g}}} + \epsilon^{-\omega\sqrt{\frac{2b-2x}{g}}}\right),$$

which is the equation to the path of the body on the plane.

PROBLEM XI.

Prove that $\Delta_x^m \Delta_y^n u_{x,y} = \Delta_y^n \Delta_x^m u_{x,y}$, *and that each expression* $= (u_{,} - 1)^m (u_{,,} - 1)^n$, *if generally* $u_{,}^r u_{,,}^s = u_{x+rh,\, y+sk}$ *where* h *and* k *are the increments of* x *and* y *respectively.*

Let $Gu_{x,y} = \phi(t)$ where y is considered constant.

$$\text{Then } G(\Delta_x^m u_{x,y}) = \left(\frac{1}{t^h} - 1\right)^m Gu_{x,y} = \left(\frac{1}{t^h} - 1\right)^m \phi(t)$$

$$= \psi(t) \text{ suppose,}$$

$$\text{and } G\Delta_y^n(\Delta_x^m u_{x,y}) = \left(\frac{1}{t^k} - 1\right)^n G(\Delta_x^m u_{x,y})$$

$$= \left(\frac{1}{t^k} - 1\right)^n \cdot \left(\frac{1}{t^h} - 1\right)^m \phi(t),$$

and $G(\Delta_x^m \Delta_y^n u_{x,y})$ equals the same expression;

$$\therefore \Delta_x^m \Delta_y^n u_{x,y} = \Delta_y^n \Delta_x^m u_{x,y}.$$

Again, $G(\Delta_x^m \Delta_y^n u_{x,y})$

$$= \left\{\frac{1}{t^{mh}} - m\cdot\frac{1}{t^{(m-1)h}} + \frac{m(m-1)}{(2)}\cdot\frac{1}{t^{(m-2)h}} - \&c.\right\}$$

$$\times \left\{\frac{1}{t^{nk}} - n\cdot\frac{1}{t^{(n-1)k}} + \frac{n(n-1)}{(2)}\cdot\frac{1}{t^{(n-2)k}} - \&c.\right\} G u_{x,y}.$$

Now the product of two terms taken out of the series

$$= \frac{(m)}{(r).(m-r)}\cdot\frac{1}{t^{(m-r)h}} \times \frac{(n)}{(s).(n-s)}\cdot\frac{1}{t^{(n-s)k}}\cdot G u_{x,y}.$$

Now $\qquad \frac{1}{t^{(m-r)h}}\cdot G u_{x,y} = G u_{x+(m-r)h,\,y}$,

and $\frac{1}{t^{(n-s)k}}\cdot\frac{1}{t^{(m-r)h}}\cdot G u_{x,y} = \frac{1}{t^{(n-s)k}}\; G u_{x+(m-r)h,\,y}$

$$= G u_{x+(m-r)h,\; y+(n-s)k}$$

$$= G(u_1^{m-r}.\,u_2^{n-s}).$$

Hence the product of any two terms equals the generating function of the product of the same two corresponding terms in $(u_1 - 1)^m . (u_2 - 1)^n$. Hence the sum of the products of every two terms or $G(\Delta_x^m \Delta_y^n u_{x,y})$ = generating function of the product of the two expressions

$$= G(u_1 - 1)^m . (u_2 - 1)^n;$$

$$\therefore\ \Delta_x^m \Delta_y^n u_{x,y} = (u_1 - 1)^m . (u_2 - 1)^n.$$

Problem 12 has been previously considered. See Problem 11 in Mr Gaskin's Paper.

Problem XIII.

A circular hoop has communicated to it a velocity of translation, parallel to a given rough inclined plane with which it is in contact, and in the downward direction; and also at the same time a velocity of rotation about its centre. Find the conditions under

which the hoop will descend to a given point on the inclined plane and then just return to the place where it set out; also point out the circumstances of the remainder of the motion.

Fig. 47. Let A be the point from which the hoop is projected with a velocity u of its centre of gravity and an angular velocity ω about that point in the direction of the arrow; $AB = b$ the distance over which the hoop moves before it commences its return; let $\tan\psi$ be the coefficient of friction, α the inclination of the plane to the horizon, a the radius of the hoop; $AD = x$, D being the line of contact at time t, and θ the angle through which a given radius of the hoop has revolved.

The equations of motion are

$$d_t^2 x = g(\sin\alpha - \tan\psi\cos\alpha) = g\sec\psi\sin(\alpha - \psi),$$

$$\frac{a^2}{2}\,.\,d_t^2\theta = -g\tan\psi\cos\alpha\,.\,a,$$

$$\text{or}\quad d_t^2\theta = -\frac{2g}{a}\tan\psi\cos\alpha.$$

The integration of these equations gives us

$$\left.\begin{aligned} d_t x &= u + g\sec\psi\sin(\alpha-\psi)\,.\,t \\ a\,d_t\theta &= a\omega - 2g\tan\psi\cos\alpha\,.\,t \end{aligned}\right\},$$

$$\text{and}\quad x = ut + \tfrac{1}{2}g\sec\psi\sin(\alpha-\psi)\,.\,t^2.$$

Now the sliding motion ceases when the velocity of the point of contact is zero, or when $d_t x = a d_t\theta$, which gives

$$gt = \frac{a\omega - u}{\sec\psi\sin(\alpha-\psi) + 2\tan\psi\cos\alpha}$$

$$= \frac{a\omega - u}{\sin\alpha + \tan\psi\cos\alpha}$$

$$= (a\omega - u)\,.\,\frac{\cos\psi}{\sin(\alpha+\psi)},$$

and $2gb = 2u \,.\, gt + \sec\psi \sin(a - \psi) \,.\, (gt)^2$

$$= \frac{2u(a\omega - u)\cos\psi}{\sin(a+\psi)} + \frac{\cos\psi \sin(a-\psi)}{\sin^2(a+\psi)} \,.\, (a\omega - u)^2,$$

or $2gb\sin^2(a+\psi) = 2u(a\omega - u)\cos\psi\sin(a+\psi)$

$$+ \cos\psi\sin(a-\psi) \,.\, (a\omega - u)^2 \text{.........} (1).$$

Also the angular velocity ω_1 at this time

$$= \omega - \frac{2\tan\psi\cos a}{\cdot} \,.\, gt;$$

$$\therefore\ a\omega_1 = a\omega - 2\tan\psi\cos a \,.\, (a\omega - u) \,.\, \frac{\cos\psi}{\sin(a+\psi)}$$

$$= \frac{a\omega\sin(a-\psi) + 2u\cos a\sin\psi}{\sin(a+\psi)}.$$

Now since in the return of the hoop towards the point of projection there is no sliding, the equation of Vis Viva gives directly an integral of the equations of motion, viz.

$$\frac{a^2}{2}(d_t\phi)^2 + (d_t y)^2 = -2gy\sin a + C,$$

$d_t y$ being the linear velocity of the centre measured from B, and $d_t\phi$ the angular velocity.

Now $\qquad y = a\phi,$

and when $\qquad y = 0, \qquad d_t\phi = \omega_1;$

$$\frac{3a^2}{2} \,.\, (d_t\phi)^2 = -2ga\phi\sin a + C,$$

$$\frac{3a^2}{2} \,.\, \omega_1^2 = C;$$

$$\frac{3a^2}{2}\{(d_t\phi)^2 - \omega_1^2\} = -2ga\phi\sin a.$$

Now suppose $d_t\phi = 0$ when $a\phi = b$, or the hoop *just* to return to the point of projection, then

$$2gb\sin\alpha = \frac{3}{2}\cdot(a\omega_1)^2,$$

$$\text{or } 4gb\sin\alpha = \frac{3\left\{a\omega\sin(\alpha-\psi) + 2u\cos\alpha\sin\psi\right\}^2}{\sin^2(\alpha+\psi)} \ldots\ldots (2).$$

The equations (1) and (2) give the values of ω and u, which the conditions of the problem require.

After this the hoop rolls down the plane, resisted by a constant friction $= \dfrac{\sin\alpha}{3} \times$ weight of the hoop, and actuated by a constant moving force $= \dfrac{2\sin\alpha}{3} \times$ weight.

Problem XIV.

On the emission theory of light, the amount of aberration is independent of the density of the medium through which the light comes to the eye, but not so as the undulatory theory.

Fig. 18. Let SAB be the path of a ray of light proceeding in vacuo from a star S, in the same time that the earth would take to move in its orbit from T to B: in this case BST is the aberration: for if the ray of light were to enter a tube AD held by an observer parallel to ST, and its motion resolved into two, one AD along the axis of the tube, the other DB parallel to the motion of the earth, the latter part would be just equal to the motion of the observer, and the ray of light would continue in the axis of the tube, and the star would appear in the direction DA. And if v be the velocity of the earth, V the velocity of light in vacuo,

$$\frac{v}{V} = \frac{BD}{AB} = \frac{\sin BAD}{\sin ADB} \qquad .(1).$$

Let μ = index of refraction of a medium of which the above tube may be composed, and AC the path of the ray through it.

Then on the emission theory (Art. Light. *Encyclop. Met.*)

$$\mu V = V' \text{ the velocity in } AC,$$

but on the undulatory theory,

$$\mu V' = V.$$

Hence, in the first case the velocity of light along CD

$$= V' \frac{CD}{AC} \quad \mu V \frac{CD}{AC} = V \frac{\sin BAD}{\sin CAD} \cdot \frac{\sin CAD}{\sin ADB}$$

$$= V \frac{\sin BAD}{\sin ADB} = v \text{ from (1).}$$

Hence, the motion of the ray of light entering the tube at A resolved as before parallel to CD will equal that of the observer: it will therefore move along the axis of the tube, and will be at C when the observer is there, the aberration BAD being the same as before.

But in the other case the velocity of light along CD $= V \frac{\sin CAD}{\sin ADB \,.\, \sin ABD}$ which does not equal v: hence it is necessary that the tube be held in a different direction, in order that the rays of light may meet the eye, or the aberration is altered by a quantity depending on $\sin CAD$, or on the density of the medium.

Problem XV.

There are four candidates for an office and four electors; each elector gives two votes, one to each of two candidates; and in the first scrutiny one candidate, viz. he who has fewest votes or none, is to be rejected. Shew that the rejected candidate may not be the one who is least approved. Also, neglecting moral causes, find the chance of its being necessary to repeat the first scrutiny because each of the four candidates has two votes.

I. Suppose the candidates to be A, B, C, D. The first three voters may divide their votes between A and D, and of the other two give the preference to C. The fourth voter divides his votes between A and B, or D and B, and C who is not the least approved may be rejected.

II. Again, suppose the 8 votes to be represented by (aa_1), (bb_1), (cc_1), (dd_1). Then twice the number of combinations of these votes taken 2 together may be represented by

$$\begin{aligned} & a\,(b + c + d + b_1 + c_1 + d_1) \\ & + b\,(a + c + d + a_1 + c_1 + d_1) \\ & + c\,(a + b + d + a_1 + b_1 + d_1) \\ & + d\,(a + b + c + a_1 + b_1 + c_1) \\ & + \quad \&c. \end{aligned}$$

Since aa_1, bb_1, &c. being the votes of one person cannot appear together. Now if one combination of two be taken out of each of the above sets (taken four together) then for each such combination of four there will be nine ways of distributing the votes: and the number of combinations of $4 = \dfrac{8 \,.\, 7 \,.\, 6 \,.\, 5}{1 \,.\, 2 \,.\, 3 \,.\, 4} = 70$.

Hence, the number of combinations of four sets containing $2 = \frac{9 \times 70}{2} = 315$, and these four sets can be varied in $1.2.3.4$ ways forming on the whole 24×315 combinations. Now each person can distribute his votes in 12 ways. Hence the required chance

$$= \frac{24 \times 315}{12.12.12.12} = \frac{35}{96}.$$

Problem XVI.

From each point of the exterior of two concentric ellipsoids, whose axes are in the same a rections, tangent planes are drawn to the surface of the interior one: shew that all the planes of contact, corresponding to the several points of the exterior surface, touch another concentric ellipsoid.

Shew also, that if tangents be drawn from each point of any curve of the second order to any other curve of the second order, however situated, the lines which join the points of contact of the pairs of tangents drawn from each point of the first curve, touch another curve of the second order.

I. Let a, b, c; α, β, γ be the semi-axes of the two ellipsoids; h, k, l the co-ordinates of a point in the exterior ellipsoid: x, y, z the co-ordinates of any point in the plane of contact corresponding to the point (hkl).

The equation to the plane of contact is

$$\frac{hx}{\alpha^2} + \frac{ky}{\beta^2} + \frac{lz}{\gamma^2} = 1\text{......}(1),$$

$$\text{and } \frac{h^2}{a^2} + \frac{k^2}{b^2} + \frac{l^2}{c^2} = 1\text{......}(2).$$

Differentiating these equations with regard to h and k, as independent variables, we have

$$\frac{x}{\alpha^2} + \frac{z}{\gamma^2} d_h l = 0$$

$$\frac{y}{\beta^2} + \frac{z}{\gamma^2} d_k l = 0$$

$$\frac{h}{a^2} + \frac{l}{c^2} d_h l = 0$$

$$\frac{k}{b^2} + \frac{l}{c^2} d_k l = 0$$

$$\therefore\ \frac{h}{l} \cdot \frac{c^2}{a^2} = \frac{x}{z} \cdot \frac{\gamma^2}{\alpha^2}$$

$$\text{and}\ \frac{k}{l} \cdot \frac{c^2}{b^2} = \frac{y}{z} \cdot \frac{\gamma^2}{\beta^2},$$

Hence
$$\left.\begin{aligned} h &= \left(\frac{a}{c}\right)^2 \cdot \frac{x}{z} \cdot \left(\frac{\gamma}{\alpha}\right)^2 l \\ k &= \left(\frac{b}{c}\right)^2 \cdot \frac{y}{z} \cdot \left(\frac{\gamma}{\beta}\right)^2 l \end{aligned}\right\},$$

and from (1)

$$\left\{\left(\frac{a}{c}\right)^2 \cdot \frac{x^2}{z} \cdot \frac{\gamma^2}{\alpha^4} + \left(\frac{b}{c}\right)^2 \cdot \frac{y^2}{z} \cdot \frac{\gamma^2}{\beta^4} + \frac{z}{\gamma^2}\right\} l = 1,$$

and from (2)

$$\frac{1}{a^2} \cdot \left\{\left(\frac{a\gamma}{c\alpha}\right)^2 \cdot \frac{x}{z}\right\}^2 + \frac{1}{b^2}\left\{\left(\frac{b\gamma}{c\beta}\right)^2 \cdot \frac{y}{z}\right\}^2 + \frac{1}{c^2} = \frac{1}{l}$$

Hence
$$\frac{\gamma^4}{c^4 z^2}\left(\frac{a^2 x^2}{\alpha^4} + \frac{b^2 y^2}{\beta^4} + \frac{c^2 z^2}{\gamma^4}\right) = \frac{1}{l^2}$$

$$= \frac{\gamma^4}{c^4 z^4}\left(\frac{a^2 x^2}{\alpha^4} + \frac{b^2 y^2}{\beta^4} + \frac{c^2 z^2}{\gamma^4}\right)^2,$$

$$\text{or } \frac{a^2x^2}{\alpha^4} + \frac{b^2y^2}{\beta^4} + \frac{c^2z^2}{\gamma^4} = 1;$$

and this is the equation to a concentric ellipsoid, touched by the planes of contact.

II. Again, let the equation to the curve of the second order, which is always touched, be

$$ay^2 + bxy + cx^2 + dy + ex + f = 0,$$

h, k the co-ordinates of a point in another curve of the second order, whose vertex is taken for origin and axis for axis of x, which satisfy the equation

$$k^2 = 2ph + nh^2 \ldots\ldots\ldots (1).$$

The equation to a chord of contact

$$(x - h) . (by + 2cx + e) + (y - k)(2ay + bx + d) = 0 \ldots (2).$$

Differentiating with regard to h, we have

$$by + 2cx + e + (2ay + bx + d) d_h k = 0,$$

$$\text{and } k d_h k = p + nh.$$

$$\text{Hence, } by + 2cx + e + \frac{p + nh}{k}(2ay + bx + d) = 0,$$

$$\text{and } \left(h + \frac{p}{n}\right)^2 = h^2 + \frac{2p}{n}h + \left(\frac{p}{n}\right)^2$$

$$= \frac{1}{n}(nh^2 + 2ph) + \left(\frac{p}{n}\right)^2$$

$$= \frac{k^2}{n} + \left(\frac{p}{n}\right)^2 \text{ from (1).}$$

Now from above

$$n\left(h + \frac{p}{n}\right)(2ay + bx + d) = -(by + 2cx + e)k;$$

$$\therefore n^2(2ay + bx + d)^2 . \left\{\frac{k^2}{n} + \left(\frac{p}{n}\right)^2\right\} = k^2(by + 2cx + e)^2;$$

$$\therefore\ k^2 \left\{(by + 2cx + e)^2 - n\,(2ay + bx + d)^2\right\}$$
$$= p^2\,(2ay + bx + d)^2 \ldots\ldots\ldots\ldots\ldots (3);$$

$$\therefore\ \left(h + \frac{p}{n}\right)^2 = \left(\frac{by + 2cx + e}{2ay + bx + d}\right)^2 \cdot \frac{k^2}{n^2};$$

$$\therefore\ \left(h + \frac{p}{n}\right)^2 \cdot \left\{(by + 2cx + e)^2 - n\,(2ay + bx + d)^2\right\}$$

$$\frac{p}{n^2}\,(by + 2cx + e)^2 \ldots\ldots\ldots\ldots (4).$$

Now from (2),

$$h\,(by + 2cx + e) + k\,.\,(2ay + bx + d)$$
$$= x\,(by + 2cx + e) + y\,(2ay + bx + d)$$
$$= -\,dy - ex - 2f.$$

From (3) and (4) we have

$$h + \frac{p}{n} = \frac{p}{n} \cdot \frac{by + 2cx + e}{\sqrt{A}},$$

$$\text{and } k = -\,p\,.\,\frac{2ay + bx + d}{\sqrt{A}},$$

since k is of different sign from $h + \dfrac{p}{n}$,

where $A = (by + 2cx + e)^2 - n(2ay + bx + d)^2$.

Hence substituting these values in the above equation, we have

$$\left.\begin{array}{l} \dfrac{p}{n} \cdot \dfrac{(by + 2cx + e)^2}{\sqrt{A}} - \dfrac{p}{n}\,(by + 2cx + e) \\ -\,p\,.\,\dfrac{(2ay + bx + d)^2}{\sqrt{A}} + dy + ex + 2f \end{array}\right\} = 0;$$

$$\therefore\ \frac{p}{n}\left\{\frac{(by + 2cx + e)^2 - n\,(2ay + bx + d)^2}{\sqrt{A}}\right\}$$

$$\frac{p}{n}(by + 2cx + e) - (dy + ex + 2f),$$

$$\text{or } \frac{p}{n} \cdot \sqrt{A} = \frac{p}{n}(by + 2cx + e) - (dy + ex + 2f);$$

$$\therefore (by + 2cx + e)^2 - n(2ay + bx + d)^2$$

$$= \left\{by + 2cx + e - \frac{n}{p}(dy + ex + 2f)\right\}^2,$$

which is the equation to a curve of the second order, to which all the chords of contact are tangents.

Problem XVII.

Two parallel indefinite plates are maintained at different constant temperatures: a thin indefinite plane lamina being placed between them so as to be parallel to each; find the temperature which it will ultimately attain, on the supposition of the space between the planes being a vacuum.

To what causes may the excessive cold of the higher regions of the atmosphere be attributed?

State the steps of the reasoning by which the temperature of that part of space in which the earth moves has been approximated to.

Since the temperature at every point of the lamina will be the same at the same time, it will suffice to consider a single particle thereof.

Let a be the distance of the lamina from one of the planes which is maintained at the temperature v, and let h be the conductivity. From the particle in question draw a perpendicular to this plane, and about the perpendicular take an annulus with radii r and $r + \delta r$.

Then supposing the heat emitted to vary as

$$\frac{\text{cos angle of emanation}}{(\text{dist.})^2}, \text{ we have}$$

$$\text{heat received from annulus} = \frac{2\pi a r \delta r}{(a^2+r^2)^{\frac{3}{2}}}\, hv \text{ ultimately,}$$

$$\text{plane} \quad = 2\pi a h v \; {}^{0}\!\int_{r}^{\infty} \frac{r}{(a^2+r^2)^{\frac{3}{2}}}$$

$$= 2\pi h v.$$

So heat received from the other plane whose temperature is v', is $2\pi h v'$, supposing the conductivity to be the same as in the former case. Now the intermediate lamina, having a *permanent* temperature, V suppose, emits as much as it receives;

$$\therefore \; 4\pi V h = 2\pi v h + 2\pi v' h,$$

$$\text{or } V = \frac{v+v'}{2}.$$

The subjects of the remainder of the problem are treated of and explained in M. Poisson's *Theorie de la Chaleur*, Art. 228, and in a supplemental Memoir read before the Academy at Paris, in 1837.

Problem XVIII.

Prove that, if at each point of space a force act which is any function of its distance from a given point A, *and* θ *be the angle at which the tangent to a point* P *of an arbitrary curve, connecting any two points* P_1, P_2 *in space, is inclined to the direction of the force* f *at* P_1, *then* $\int f \cos\theta\, ds$, *from* P_1 *to* P_2 *depends only on the distances* AP_1, AP_2.

Assuming that the law of force in the action of the particles of the magnetic fluid is that of the inverse square of the distance, shew that the existence of two points on the earth's surface, where the total horizontal force vanishes, and the total intensity is a maximum for both, or a minimum for both, necessitates the existence of a third point where the total horizontal force vanishes.

Let $\phi(r^2)$ = force at the point (x, y, z), the fixed point A being the or

$$\text{Then force parallel to } x = \frac{x}{r}\,\phi(r^2)$$

$$= d_x \int_r \phi(r^2).$$

$$\text{Since}\quad \left.\begin{array}{r} x^2 + y^2 + z^2 = r^2 \\ \text{and} \quad \therefore\ x = r d_x r \end{array}\right\},$$

$$\text{or}\quad X = d_x V,$$

$$\text{if}\quad V = \int_r \phi(r^2),$$

and X represent the force parallel to x.

$$\text{Similarly,}\qquad Y = d_y V, \qquad Z = d_z V.$$

$$\text{Now}\qquad dV = d_x V dx + d_y V dy + d_z V dz$$

$$= ds\,(X d_s x + Y d_s y + Z d_s z)$$

$$= ds\, f \cos\theta;$$

$$\therefore\ \int f \cos\theta\, ds = V = \int_r \phi(r^2) = \psi(r^2),$$

suppose, and the value of this integral taking between the limits $r = AP_1$, $r = AP_2$, is $\psi(AP_2^2) - \psi(AP_1^2)$ which depends only on the distances AP_1, AP_2. For the solution of the remainder of the question, and for other subjects connected with Terrestrial Magnetism, the reader is referred to a Memoir of Gauss, in Taylor's *Scientific Memoirs*. 1840.

Problem XIX.

If a beam hang vertically by means of a string attached to its upper extremity and to a fixed point, shew that there are two ways in which the beam can oscillate so that each point of it comes to the vertical position of rest at the same instant, and find the respective times of small oscillations of this nature.

Fig. 52. Let the figure represent the positions of the beam and string at the time t: x, y the vertical and horizontal co-ordinates of the centre of gravity of the beam referred to O as origin, ϕ its inclination to the vertical, and θ the inclination of the string (b) to the vertical: $2a$ the length of the beam, m its mass, mT the tension of the string, (k) the radius of the gyration about an axis through G and perpendicular to the rod.

The equations of motion are

$$d_t^2 x = g - T\cos\theta,$$

$$d_t^2 y = -T\sin\theta,$$

$$k^2 d_t^2\phi = -Ta\sin(\phi+\theta),$$

$$x = b\cos\theta + a\cos\phi,\quad y = b\sin\theta - a\sin\phi.$$

Eliminating T, the two final equations are

$$k^2\cos\theta\, d_t^2\phi = -a\sin(\phi+\theta)\,\{g + b\cos\theta\,(d_t\theta)^2$$

$$+ a\cos\phi\,(d_t\phi)^2 + b\sin\theta\, d_t^2\theta + a\sin\phi\, d_t^2\phi\},$$

$$k^2\sin\theta\, d_t^2\phi = a\sin(\phi+\theta)\,\{b\cos\theta\, d_t^2\theta - a\cos\phi\, d_t^2\phi$$

$$- b\sin\theta\,(d_t\theta)^2 - a\sin\phi\,(d_t\phi)^2\}.$$

However, when the oscillations are small, these equations are reducible to

$$\left.\begin{aligned} d_t^2 y + g\theta &= 0 \\ k^2 d_t^2\phi + ga(\phi+\theta) &= 0 \end{aligned}\right\};$$

$$y = b\theta - a\phi,$$

which may be derived from the original equations by considering the tension as equal to the weight, and omitting quantities of a higher order than θ and ϕ.

Eliminating θ they become

$$\left.\begin{aligned} d_t^2 y + \frac{g}{b} y + \frac{ga}{b} \phi = 0 \\ k^2 d_t^2 \phi + ga\left(1 + \frac{a}{b}\right)\phi + g\frac{a}{b} y = 0 \end{aligned}\right\} \dots\dots\dots (1).$$

To solve these equations, (Poisson's *Dynamique*, Art. 546) we observe that they are satisfied by

$$y = RN \sin (t\sqrt{\rho} - r), \quad \phi = RN' \sin (t\sqrt{\rho} - r),$$

R and r being arbitrary constants, ρ, N, N' constants to be determined.

Substituting these in equations (1) we have for determining $\frac{N}{N'}$ and ρ.

$$\left.\begin{aligned} -k^2 N'\rho + ga\left(1 + \frac{a}{b}\right) N' + g\frac{a}{b} N = 0 \\ -N\rho + \frac{g}{k} N + g \cdot \frac{a}{k} N' = 0 \end{aligned}\right|$$

Hence $\dfrac{ga}{b\rho - g} = \dfrac{N}{N'} = \dfrac{k^2 \rho b - ga(a+b)}{ga}$

Hence we have the quadratic equation

$$k^2 b^2 \rho^2 - gb(k^2 + a^2 + ab)\rho + g^2 ab = 0,$$

which gives two values of ρ. Now y and ϕ go through all their changes in value, while $t\sqrt{\rho}$ is increased by π, or while t is increased by $\dfrac{\pi}{\sqrt{\rho}}$, which is hence the time of oscillation: and as y and ϕ vanish together, each point of the beam comes to its vertical position of rest

at the same instant. As there are two values of ρ $(\rho_1\ \rho_2)$, there are two ways in which the oscillations can take place, their respective times being $\dfrac{\pi}{\sqrt{\rho_1}}$ and $\dfrac{\pi}{\sqrt{\rho_2}}$: they are indicated in the figure.

If $\dfrac{y}{\phi} = \dfrac{N}{N'} = 0$, that is, if the centre of gravity of the beam be in the vertical,

$$k^2 \rho b = ga(a + b),$$

and the time of an oscillation $= \dfrac{\pi}{\sqrt{\rho}}$

$$\frac{\pi k}{\sqrt{ga}} \cdot \sqrt{\frac{b}{a+b}}.$$

PROBLEM XX.

Find an expression for the time of a small oscillation of any body about a fixed horizontal axis, the body being partly immersed in a fluid, and deduce from it the times of oscillation (1) *when the body floats freely and moves about its centre of gravity,* (2) *when it is wholly immersed. The resistance of the medium is not to be taken into account.*

Fig. 49. Let the plane of the paper cut the axis perpendicularly in O, G the centre of gravity of the solid, H that of the fluid displaced by it, whose volume is V and density ρ, M the point in which the vertical, through the centre of gravity of the fluid displaced, meets OG, when that line makes a small angle θ with the vertical. M the mass of the solid, k the radius of gyration of the solid about the fixed axis, k_1 that of the plane in which the surface of the fluid intersects the solid, about an axis through its centre of gravity.

The equation of the motion of rotation is

$$Mk^2 d_t^2\theta = -(Mg\,.\,OG - g\rho V\,.\,OM)\sin\theta,$$

or $$d_t^2\theta + g\frac{M\,.\,OG - \rho V\,.\,OM}{Mk^2}\theta = 0,$$ since θ is small.

Hence the time of a small oscillation

$$= \pi\sqrt{\frac{Mk^2}{g(M\,.\,OG - \rho V\,.\,OM)}}.$$

Now in the first case, since the solid floats freel $M = \rho V$; and since the rotation is about G, $OG =$ Also, as the equilibrium must be supposed stable, l n this case, lies above the axis, and we must wri $-GM$ for OM. Hence the time of a small oscillation

$$= \pi\sqrt{\frac{k^2}{g\,.\,GM}}$$

$$= \frac{\pi k}{\sqrt{g\left(\frac{k_1^2 A}{V} - GH\right)}},$$

A being the area of the plane of floatation.

In the second case in which the solid is wholly immersed, $GH = 0$ and $GM = 0$, and the time of a small oscillation

$$= \pi\sqrt{\frac{Mk^2}{Mg\,.\,OG - g\rho V(OG + MG)}}$$

$$= \pi\sqrt{\frac{Mk^2}{g(M - \rho V)OG}}.$$

NOTE I.

WHEN a solid is at rest in a fluid whose density varies as the depth, then the density of the fluid at the depth of the centre of gravity of the solid is the same as that of the solid. To prove this, let Z = depth of the centre of gravity of the solid beneath the surface of the fluid; p = pressure at depth Z; x, y, z the co-ordinates of any point of the surface of the solid.

Then $$dp = \mu g z dz,$$

and pressure $= \dfrac{\mu g z^2}{2}$

$$= \frac{\mu g z^2}{2} \cdot \frac{1}{\sqrt{1 + (d_x z)^2 + (d_y z)^2}} \text{ parallel to the axis of } z.$$

and vertical pressure on an element of the surface

$$= \frac{\mu g z^2}{2} \cdot \frac{\delta x \delta y \sqrt{1 + (d_x z)^2 + (d_y z)^2}}{\sqrt{1 + (d_x z)^2 + (d_y z)^2}}$$

$$= \frac{\mu g z^2}{2} \delta x \delta y, \text{ ultimately};$$

therefore whole vertical pressure at the surface

$$= \frac{\mu g}{2} \int_x \int_y z^2 = \mu g \int_x \int_y \int_z z,$$

the limits being taken so as to include the whole surface of the solid

Now if V = volume of the solid,

ρ = its density.

Then $$Z . V = \int_x \int_y \int_z z;$$

$\therefore$ $g \rho V$ = weight of the solid

$$= \mu g Z . V;$$

$$\therefore \rho = \mu Z,$$

whence the proposition

N B. The solid is supposed wholly immersed

NOTE II. (Page 47.)

Let the principal plane of the cone be the plane of xz, then since for every positive value of y there will be an equal negative value, the equation to the cone is

$$ax^2 + by^2 + cz^2 + c'xz = 0.$$

Let the equations to the circular section be

$$z = Ax + By + C,$$

$$(x - m)^2 + (y - n)^2 + (z - p)^2 = r^2.$$

Then the curves

$$ax^2 + by^2 + c(Ax + By + C)^2 + c'x(Ax + By + C) = 0,$$

$$\text{and } (x - m)^2 + (y - n)^2 + (Ax + By + C - p)^2 = r^2,$$

are identical. Hence equating coefficients of xy,

$$2cAB + c'B = q \cdot 2AB,$$

which is satisfied by $B = 0$. The other relation gives no result; hence the plane of the section is perpendicular to the principal plane.

NOTE III.

As the equation to the path of a projectile and to an orbit described about a centre of force, which have been used, may be obtained from the Principle of Least Action, we shall so derive them in this note.

Suppose x, y to be the co-ordinates of a particle at time t, acted on by the forces X, Y parallel to the axes; v its velocity.

Then $$v^2 = 2\int (X\,dx + Y\,dy),$$

and by the Principle of Least Action

$$\int\{\sqrt{2\int(X\,dx + Y\,dy)}\,ds\} = \text{minimum}.$$

I. Suppose $X = 0$, $Y = -g$, $p = d_x y$.

Then $$\int_x \sqrt{C - 2gy}\,.\,\sqrt{1 + p^2} = \text{minimum},$$

or if u be the value of v when $t = 0$,

$$\int_x \sqrt{u^2 - 2gy}\,.\,\sqrt{1 + p^2} = \text{minimum}.$$

Here using the notation of the Calculus of Variations, we have

$$V = \sqrt{u^2 - 2gy}\,.\,\sqrt{1 + p^2} = Pp + C$$

$$= \frac{p^2\sqrt{u^2 - 2gy}}{\sqrt{1 + p^2}} + C;$$

$$\therefore \sqrt{u^2 - 2gy} = C\sqrt{1 + p^2},$$

and suppose when $y = 0$, $\quad p = \tan\alpha$,

$$\text{then } \sqrt{u^2 - 2gy} = u\cos\alpha\,.\,\sqrt{1 + p^2};$$

$$\therefore \frac{d_y x}{u\cos\alpha} = \frac{1}{\sqrt{u^2\sin^2\alpha - 2gy}};$$

$$\therefore \frac{x}{u\cos u} + C = -\frac{1}{g}\sqrt{u^2\sin^2\alpha - 2gy},$$

and suppose when $x = 0$, $\quad y = 0$,

$$\text{then } \frac{u\sin\alpha}{u\cos\alpha} \qquad \frac{\sqrt{u^2\sin^2\alpha - 2gy}}{g}$$

$$\therefore y = x\tan\alpha - \frac{gx^2}{2u^2\cos^2\alpha}$$

which is the equation to the path of a projectile.